Hydrodynamics for symmetric exclusion in contact with reservoirs

Patrícia Gonçalves

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Contents

Chapter 1

Introduction

These notes have been written based on material of the articles [1], [2] and [3] which was presented on a mini-course that the author gave while visiting Institut Henri Poincaré in Paris in May 2017 for the trimester "Stochastic dynamics out of equilibrium" that held from the 3rd of April to the 7th of July. The slides and the videos of the mini-course can be seen in
`https://indico.math.cnrs.fr/event/844/page/5`.

The content of the notes is to explain how to derive partial differential equations with different types of boundary conditions from varied underlying microscopic stochastic dynamics. In the next coming chapters we consider a macroscopic space which is the interval $[0,1]$ and which is discretized according to a scaling parameter N giving rise to N intervals of size $1/N$. To each $q \in [0,1]$ belonging to the interval $[i/n, i+1/n)$ we associate to it the point i/n and in the discrete set of points $\{1,...,i,...N-1\}$ we will define a microscopic dynamics of exclusion type which is Markovian. The discrete set of points $\{1,...,i,...N-1\}$ will be called the bulk and to it we add two extra points $x=0$ and $x=N$ which will act as reservoirs. The exclusion dynamics ensures that there is at most one particle per site in the bulk and the Markovian dynamics comes from the fact that each particle waits for rings of random clocks exponentially distributed and independent, after which the particle jumps from a site x in the bulk to another site y in the bulk according to a probability transition rate $p : \mathbb{Z} \times \mathbb{Z} \to [0,1]$, or the particle leaves the system through one of the reservoirs. The reservoirs will be regulated by a parameter which has the capacity to slow or fast the boundary dynamics. More precisely, particles can be injected in the bulk from the site $x=0$ (resp. $x=N$) to the site y at rate $\alpha\kappa N^{-\theta}p(y)$ (resp. $\beta\kappa N^{-\theta}p(N-y)$) and can be removed from the bulk at the site y to the site $x=0$ (resp. $x=N$)

at rate $(1-\alpha)\kappa N^{-\theta}p(y)$ (resp. $(1-\beta)\kappa N^{-\theta}p(N-y)$). Above, $\alpha, \beta \in [0,1]$, $\theta \in \mathbb{R}$ and $\kappa > 0$. The goal in these notes is to derive the partial differential equations which describe the space-time evolution of the density of particles in the system. These equations will have boundary conditions which will depend on the strength of the boundary dynamics, namely, the parameter θ.

The goal is to analyse which type of boundary conditions we can get and what is their dependence on the strength of the reservoirs. For that purpose, we split these notes into two main chapters to distinguish the case in which jumps are nearest-neighbor or not. Therefore in Chapter 2, we consider the dynamics described above but with $p : \mathbb{Z}\times\mathbb{Z} \to [0,1]$ which satifies $p(x,y) = p(y-x) = 0$ if $|x-y| > 1$, $p(0) = 0$ so that $p(1) = p(-1) = \frac{1}{2}$. This means that in the bulk particles can jump to their nearest-neighbors and particles can be injected/removed in the bulk/from the bulk through the sites $x = 1$ or $x = N-1$. For these models we will derive the heat equation with three different types of boundary conditions: non-homogenenous Dirichlet boundary conditions when the reservoirs are fast (which corresponds to $\theta < 1$) and Neumann boundary conditions when the reservoirs are slow (which corresponds to $\theta > 1$). Linking the aforementioned two types of boundary conditions, for a particular strength of the boundary dynamics (which corresponds to $\theta = 1$), we will derive the heat equation with a type of linear Robin boundary conditions.

In Chapter 3, we will consider the dynamics described above, but allowing long jumps given by a probability transition rate $p : \mathbb{Z} \times \digamma \to [0,1]$ such that $p(x,y) = p(y-x)$, which is symmetric, namely $p(y-x) = p(x-y)$, and we will distinguish two cases: the first one where $p(\cdot)$ has finite variance and then the case where $p(\cdot)$ has infinite variance. In the first case, we will obtain an extension of the results of the model with only nearest-neighbor jumps, that is we will derive the heat equation with the three types of boundary conditions mentioned above but for a certain choice of the transition probability two new regimes appear when the reservoirs are fast, namely, a reaction-diffusion equation and a reaction equation, both endowed with non-homogeneous Dirichlet boundary conditions. In the case where $p(\cdot)$ has infinite variance and for a particular strength of the reservoirs (which corresponds to $\theta = 0$), we will derive a collection of fractional reaction-diffusion equations with non-homogeneous Dirichlet boundary conditions. For the interested reader we note that when $p(\cdot)$ has infinite variance and when the strength of the reservoirs is slow (which corresponds to $\theta > 0$), we cannot say anything about the equation nor its boundary conditions. In [2] a similar model has been studied and some conjectures have been presented in the case where the reservoirs are slow. We believe that the same conjecture should be true for this model, but we leave this for a future problem to look at. We also note that it would be very interesting to consider other types of boundary dynamics or even more general type of bulk dynamics than the exclusion dynamics in order to obtain other partial differential equations with various boundary conditions.

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Chapter 2

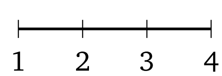
Symmetric simple exclusion in contact with reservoirs

2.1 The models

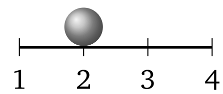
In this section we describe the collection of models that we are going to consider in these notes. First we start by fixing the notation which fits all the models and then we particularize our choice of the parameters in such a way that we treat each model, with its special features, separately.

For that purpose, we denote by N a scaling parameter, which will be taken to infinity later on. For $N \geq 2$ we denote by $\Lambda_N = \{1, \ldots, N-1\}$ the discrete set of points to which we call the bulk.

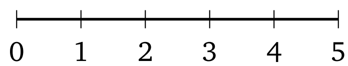
The exclusion process in contact with stochastic reservoirs is a Markov process, that we denote by $\{\eta_t : t \geq 0\}$, which has state space $\Omega_N := \{0,1\}^{\Lambda_N}$. The configurations of the state space Ω_N are denoted by η, so that for $x \in \Lambda_N$, $\eta(x) = 0$ means that the site x is vacant while $\eta(x) = 1$ means that the site x is occupied. For an illustration of the dynamics let us first take $N = 5$ so that the bulk is the discrete set of points $\{1, 2, 3, 4\}$:



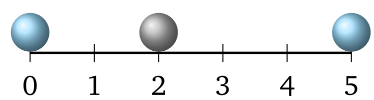
Now, to describe a possible initial configuration we can do the following. Toss a coin, if we get head we put a particle at the site 1 and if we get a tail we leave it empty. Repeat this for each site of the discrete set Λ_N and suppose that we got at the end to the configuration $\eta_0 = (0, 1, 0, 0)$ which can be represented as:



Now, we start to particularize our choice for the dynamics. We are going to add one reservoir at each end point of the bulk. This means that in our construction, we add the points $x = 0$ and $x = N$ to the bulk. Going back to the picture above, this means that we have now the set $\{0, 1, 2, 3, 4, 5\}$ where particles can be placed, but the sites $x = 0$ and $x = 5$ will act as reservoirs.



Note that the bulk stays unchanged, the role of the boundary points $\{0, N\}$ is to allow particles to get in and out of the bulk. So, for example, in the initial configuration given above, now we have the sites $x = 0$ and $x = N$ occupied, representing the fact that in $x = 0$ and $x = N$ there are particles that can enter to the bulk and that can be removed from the bulk to the reservoirs.



Now we describe the time between jumps. For that purpose, for each pair of sites (x, y) we associate a Poisson process of intensity $p(x, y) = p(y - x)$. The Poisson processes associated to different bonds are independent. Note that the bonds in the bulk are not oriented. In the first dynamics that we are describing, we consider $p(y-x) = 0$ if $|x-y| > 1$, $p(1) = p(-1) = \frac{1}{2}$ so that jumps can only occur to a nearest-neighbour position and for that reason the exclusion process coins the name *simple exclusion process*. At the boundary points we associate two Poisson processes to each bond containing a boundary point. More precisely, to the bond $\{0, 1\}$ (resp. $\{1, 0\}$) we associate a Poisson process of intensity $\alpha\kappa N^{-\theta}$ (resp. $(1-\alpha)\kappa N^{-\theta}$) and to the bond $\{N-1, N\}$ (resp. $\{N, N-1\}$) we associate a Poisson process of intensity $(1-\beta)\kappa N^{-\theta}$ (resp. $\beta\kappa N^{-\theta}$). Above we fix the parameters $\alpha, \beta \in [0, 1]$, $\theta \in \mathbb{R}$ and $\kappa > 0$. The role of the parameter θ is to regulate the slowness/fastness of the reservoirs. If $\theta > 0$ and θ increases then the reservoirs are slower and if $\theta < 0$ and θ decreases then the reservoirs are faster.

We remark that another interpretation of the previous dynamics at the boundary could be given as follows. Particles can either be created or annihilated at the sites $x = 1$ and $x = N - 1$ according to the following rates:

- at site $x = 1$:
  - creation rate $\alpha\kappa N^{-\theta}$,
  - annihilation rate $(1-\alpha)\kappa N^{-\theta}$,
- at site $x = N-1$:
  - creation rate $\beta\kappa N^{-\theta}$,
  - annihilation rate $(1-\beta)\kappa N^{-\theta}$.

Note that in any case, the exclusion rule has to be respected. At most one particle is allowed at each site of the bulk (recall that the state space is $\{0,1\}^{\Lambda_N}$) so that particles can only be created (resp. removed) at the sites $x = 1$ or $x = N-1$ if the corresponding site is empty (resp. occupied), otherwise nothing happens.

Before we proceed let us see an illustration of a possible realization of the Poisson processes as given in the figure below.

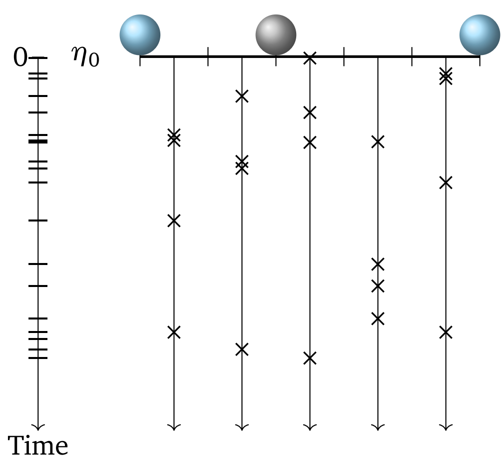


Figure 2.1: Marks of Poisson processes.

At the right hand side in Figure 2.1 we represent by "×" each mark of a possible realization of the Poisson processes associated to the bonds. At the left hand side we put an arrow going down which is representing the evolution of time and each sign "−" means that a clock has rung according to some Poisson clock, so that at the corresponding time, a jump from a particle might have occurred.

We note that in Figure 2.1 we did not distinguish the marks of the Poisson processes associated to the oriented bonds at the boundary because we believe that it is simpler to analyse the dynamics at the boundary by allowing particles to get in or get out according to the Poisson marks but also taking into account the exclusion rule.

In order to give an example, let us see now all the configurations that we obtain starting the dynamics from the configuration $\eta_0 = (0,1,0,0)$ represented above and the realization of the Poisson processes given in the previous figure.

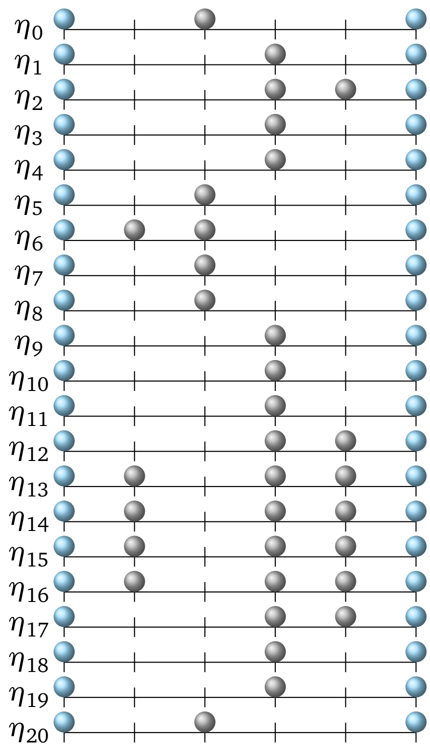


Figure 2.2: Possible configurations starting from $(0,1,0,0)$

By abuse of notation, in the Figure at the left hand side, we numbered the configurations that we obtained by the number of the marks of the Poisson processes (which in the example are equal to 20) just to make the presentation simple. We note that the configurations are indexed by time t which is continuous and not discrete. Note that the difference between $\eta_0 = (0,1,0,0)$ and $\eta_1 = (0,0,1,0)$ is only at two sites (this is always the case when we compare two configurations which differ by a jump of a particle in the bulk, *a jump in the bulk affects the occupation variables at two sites*) and η_1 is obtained from η_0 by shifting the particle at the site 2 in η_0 to the site 3. This is a consequence of the fact that the first mark of the Poisson process that occurs is associated to the bond $\{2,3\}$ and that in η_0 there us a particle at the site 2. The next mark we see is associated to the bond $\{4,5\}$ and since in $\eta_1 = (0,0,1,0)$ there is no particle at the site $x = 4$, a particle is injected in the bulk at the site 4, giving rise to $\eta_2 = (0,0,1,1)$ and so on. Note that the boundary dynamics only changes the configuration at one site.

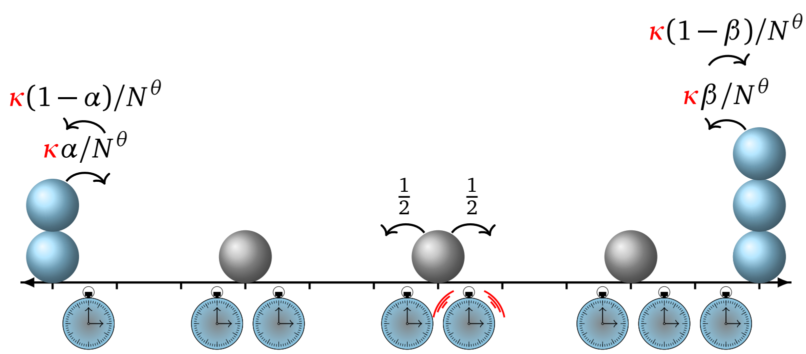
We also note that the ring of a clock does not imply that the configuration of the system has changed. In the example above $\eta_3 = \eta_4 = (0,0,1,0)$ since the corresponding Poisson mark is associated to the bond $\{1,2\}$ and since both sites $x=1$ and $x=2$ are empty, nothing happens and particles wait a new ring of a clock.

The first dynamics that we are going to consider in these notes, and which is described in this chapter is completely characterized by now, but we note that in Chapter 3 we are going to generalize the previous dynamics by allowing particles to give long jumps according to some probability transition rate $p : \mathbb{Z}\times\mathbb{Z} \to [0,1]$ such that $p(x,y) = p(y-x)$ and which is symmetric, that is $p(y-x) = p(x-y)$. In the latter dynamics, there is only one reservoir at each end point of the bulk but can particles can be injected from them to any site of the bulk or they can be removed from any site of the bulk to one of the reservoirs. We will distinguish two cases: when $p(\cdot)$ has finite variance and when $p(\cdot)$ has infinite variance.

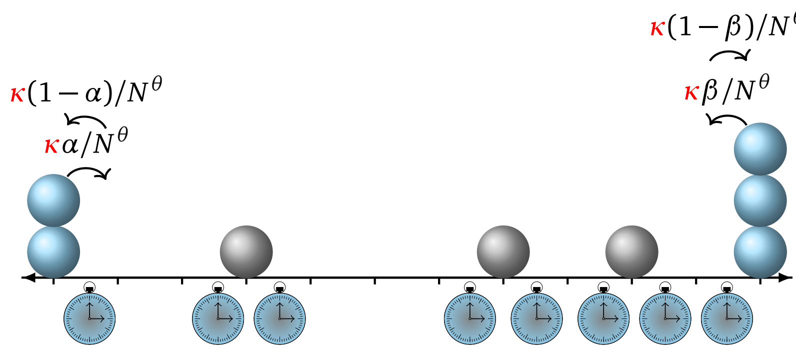
2.2 Illustration of the dynamics

In this section we draw some pictures to illustrate more easily the dynamics that we defined in the previous subsection. The particles at the bulk are coloured in gray and the particles at the two reservoirs are coloured in blue. We also added the clocks only at the bonds where there are particles but we note that the clocks are present in all bonds of the form $\{x, x+1\}$. Whenever there is a ring of a clock we see some red lines on top of the corresponding clock and the jump rates are indicated above the corresponding jumps which are represented by arrows.

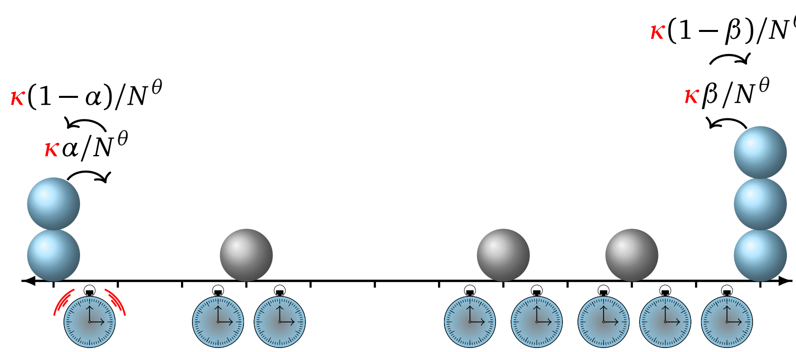
In the first picture below, we take $N = 11$ and the initial configuration is $\eta_0 = (0, 1, 0, 0, 0, 1, 0, 0, 1, 0)$. Note that this initial configuration changes only if one of the clocks associated to bonds containing the sites $x = 2, 6, 9$ rings (which makes the corresponding particle to displace one position to the left or right of it) or if the clocks at the boundary sites $x = 0$ (resp. $x = 11$) ring (which makes a particle get into the system at the site $x = 1$ (resp. $x = 10$).



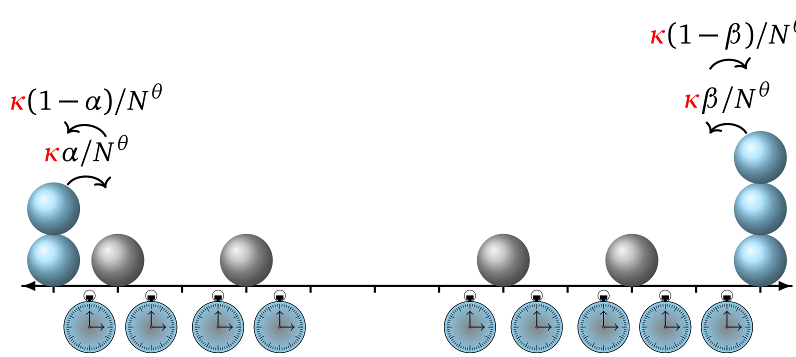
Suppose that the first clock to ring is associated to the bond $\{6, 7\}$. Since there is a particle at the site $x = 6$ it jumps to the site $x = 7$ with rate $1/2$. See the figure below.



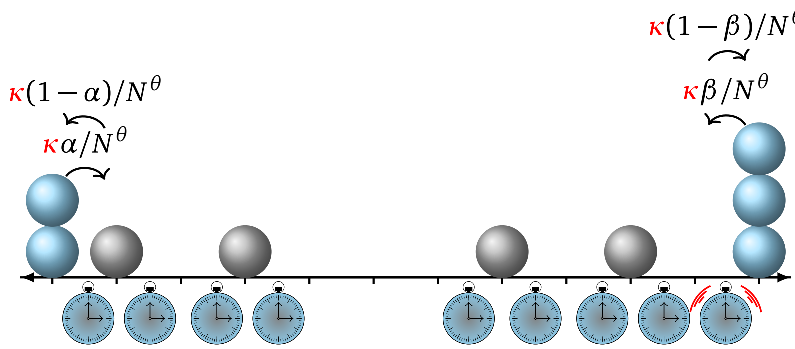
Now let us suppose that the next clock to ring is associated to the oriented bond $\{0,1\}$.



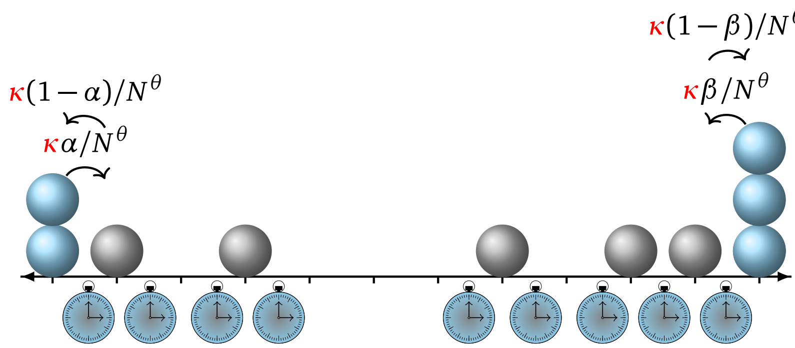
Since there is no particle at the site $x=1$, a particle is injected into the system at the site $x=1$ with rate $\alpha\kappa N^{-\theta}$. See the figure below.



Finally let us suppose that the next clock to ring is associated to the oriented bond $\{N,N-1\}$.



Since there is no particle at the site $x = N-1$, a particle is injected into the system at the site $x = N-1$ with rate $\beta\kappa N^{-\theta}$. See the figure below.



2.3 Infinitesimal generator

The dynamics described above is Markovian and can be completely characterized by mean of its infinitesimal generator. The Markov process $\{\eta_t \,:\, t \geq 0\}$ whose dynamics we have just defined has infinitesimal generator denoted by $\mathscr{L}_N$ which is expressed as

$$\mathscr{L}_N = \mathscr{L}_{N,0} + \mathscr{L}_{N,b}, \tag{2.3.1}$$

where $\mathscr{L}_{N,0}$ and $\mathscr{L}_{N,b}$ are given on functions $f : \Omega_N \to \mathbb{R}$ by

$$(\mathscr{L}_{N,0} f)(\eta) = \sum_{x=1}^{N-2} \frac{1}{2}\Big(f(\eta^{x,x+1}) - f(\eta)\Big),$$

$$\mathscr{L}_{N,b} = \mathscr{L}_{N,b}^{1} + \mathscr{L}_{N,b}^{N-1}, \tag{2.3.2}$$

where for $x \in \{1, N-1\}$

$$(\mathscr{L}_{N,b}^{x} f)(\eta) \;=\; \frac{\kappa}{N^{\theta}} c_x(\eta, r(x))\Big(f(\eta^x) - f(\eta)\Big),$$

$r(1) = \alpha$ and $r(N-1) = \beta$,

$$(\eta^{x,y})(z) = \begin{cases} \eta(z), \; z \neq x, y, \\ \eta(y), \; z = x, \\ \eta(x), \; z = y \end{cases}, \quad (\eta^x)(z) = \begin{cases} \eta(z), \; z \neq x, \\ 1-\eta(x), \; z = x, \end{cases} \tag{2.3.3}$$

and for $x \in \Lambda_N$ and $y \in \{0, N\}$

$$c_x(\eta; r(x)) := \frac{1}{2}\left[\eta(x)(1-r(x)) + (1-\eta(x))r(x)\right]. \tag{2.3.4}$$

Note that the generator above splits into the sum of the generator $\mathscr{L}_{N,0}$ (which is related to the jumps in the bulk) and $\mathscr{L}_{N,b}$ (which is related to the jumps from the boundary or from the reservoirs). We will refer to the first one as the *exchange dynamics* and the latter one as the *flip dynamics*, because in $\mathscr{L}_{N,0}$ we exchange the occupation variables $\eta(x)$ and $\eta(x+1)$ and in $\mathscr{L}_{N,b}^{x}$ we flip the value of the occupation variable at $\eta(x)$.

We consider the Markov process speeded up in the time scale $\Theta(N)$ and we note that the process $\{\eta_{t\Theta(N)} : t \geq 0\}$ has infinitesimal generator given by $\Theta(N)\mathscr{L}_N$. To see this relation, let $\tilde{\mathscr{L}}_N$ be the generator of the process $\{\eta_{t\Theta(N)} : t \geq 0\}$. By definition, for $f : \Omega_N \to \mathbb{R}$, we have that

$$\tilde{\mathscr{L}}_N f = \lim_{s \to 0} \frac{\tilde{S}_s f - f}{s}, \tag{2.3.5}$$

where $\tilde{S}_s := S_{s\Theta(N)}$ is the semigroup associated to $\tilde{\mathscr{L}}_N$ and S_s is the semigroup associated to $\mathscr{L}_N$. Then,

$$\Theta(N)\mathscr{L}_N f = \lim_{t \to 0} \Theta(N)\frac{S_t f - f}{t} = \lim_{s \to 0} \Theta(N)\frac{S_{s\Theta(N)} f - f}{s\Theta(N)} = \tilde{\mathscr{L}}_N f, \tag{2.3.6}$$

we conclude that $\tilde{\mathscr{L}}_N := \Theta(N)\mathscr{L}_N$.

We note that $\eta_{t\theta(N)}$ depends on α, β, θ and κ but we will omit these indexes in order to simplify notation. Fix $T > 0$ and $\theta \in \mathbb{R}$. We denote by $\mathbb{P}_{\mu_N}$ the probability measure in the Skorohod space $\mathscr{D}([0, T], \Omega_N)$ induced by the Markov process $\{\eta_{t\Theta(N)} : t \geq 0\}$ and the initial probability measure μ_N and we denote by $\mathbb{E}_{\mu_N}$ the expectation with respect to $\mathbb{P}_{\mu_N}$.

Our goal in these notes is to analyse the impact of changing the strength of the reservoirs (by changing the value of θ) on the macroscopic behaviour of the system. More precisely, we want to obtain the hydrodynamic equations of the process which will have different boundary conditions depending on the range of the parameter θ which rules the strength of the reservoirs. Before proceeding, in the next subsection we analyse the invariant measures for this model.

2.4 Stationary measures

For $\rho \in (0,1)$ we denote by ν_ρ^N the Bernoulli product measure in Ω_N with density ρ, that is, for $x \in \Lambda_N$:

$$\nu_\rho^N\{\eta : \eta(x) = 1\} = \rho. \tag{2.4.1}$$

According to this measure the occupation variables $\{\eta(x)\}_{x\in\Lambda_N}$ are independent and for each $x \in \Lambda_N$ the random variable $\eta(x)$ has Bernoulli distribution of parameter ρ. When we restrict the parameters α and β such that $\alpha = \beta = \rho$, then these measures are invariant for the dynamics described above. In fact, a stronger result is true, see the next lemma where we prove that that these measures are reversible.

Lemma 2.4.1. *For $\alpha = \beta = \rho$ the Bernoulli product measures ν_ρ^N are reversible.*

Proof. Fix two functions $f, g : \Omega_N \to \mathbb{R}$. To prove the lemma, we need to show that

$$\int_{\Omega_N} g(\eta)\mathscr{L}_N f(\eta)d\,\nu_\rho^N = \int_{\Omega_N} f(\eta)\mathscr{L}_N g(\eta)d\,\nu_\rho^N. \tag{2.4.2}$$

Let us start with the exchange dynamics given by $\mathscr{L}_(N,0)$. In this case we need to check that

$$\sum_{x\in\Lambda_N}\int_{\Omega_N} g(\eta)(f(\eta^{x,x+1}) - f(\eta))d\,\nu_\rho^N = \sum_{x\in\Lambda_N}\int_{\Omega_N} f(\eta)(g(\eta^{x,x+1}) - g(\eta))d\,\nu_\rho^N.$$

For that purpose note that, for fixed $x \in \Lambda_N$ and performing a change of variables $\xi = \eta^{x,x+1}$, we have that

$$\begin{aligned}\int_{\Omega_N} g(\eta)f(\eta^{x,x+1})d\,\nu_\rho^N &= \sum_{\eta\in\Omega_N} g(\eta)f(\eta^{x,x+1})\,\nu_\rho^N(\eta)\\ &= \sum_{\xi\in\Omega_N} g(\xi^{x,x+1})f(\xi)\frac{\nu_\rho^N(\xi^{x,x+1})}{\nu_\rho^N(\xi)}\,\nu_\rho^N(\xi).\end{aligned}$$

Now note that
$$\nu_\rho^N(\xi) = \prod_{x\in\Lambda_N} \rho^{\xi(x)}(1-\rho)^{1-\xi(x)}$$
so that

- if $\xi(x)=1$ and $\xi(x+1)=0$, denoting by $\tilde{\xi}$ the configuration ξ removing its values at x and $x+1$ so that $\xi=(\tilde{\xi},\xi(x),\xi(x+1))$, then $\nu_\rho^N(\xi)=\nu_\rho^N(\tilde{\xi})\rho(1-\rho)$ and $\nu_\rho^N(\xi^{x,x+1})=\nu_\rho^N(\tilde{\xi})(1-\rho)\rho$, so that
$$\frac{\nu_\rho^N(\xi^{x,x+1})}{\nu_\rho^N(\xi)} = 1. \tag{2.4.3}$$
- if $\xi(x)=0$ and $\xi(x+1)=1$, then $\nu_\rho^N(\xi)=\nu_\rho^N(\tilde{\xi})(1-\rho)\rho$ and $\nu_\rho^N(\xi^{x,x+1})=\nu_\rho^N(\tilde{\xi})\rho(1-\rho)$, so that (2.4.3) is also true.

Therefore, we obtain that
$$\int_{\Omega_N} g(\eta)f(\eta^{x,x+1})d\,\nu_\rho^N = \sum_{\xi\in\Omega_N} g(\xi^{x,x+1})f(\xi)\nu_\rho^N(\xi) = \int_{\Omega_N} g(\eta^{x,x+1})f(\eta)d\,\nu_\rho^N.$$
which proves (2.4.2) for $\mathscr{L}_{N,0}$. For the flip dynamics given by $\mathscr{L}_{N,b}$ we note, for the left boundary, that
$$\begin{aligned}&\int_{\Omega_N} g(\eta)c_1(\eta,\alpha)f(\eta^1)d\,\nu_\rho^N\\ &\quad= \sum_{\eta\in\Omega_N} g(\eta)(1-\eta(1))\alpha f(\eta^1)\nu_\rho^N(\eta) + \sum_{\eta\in\Omega_N} g(\eta)(1-\alpha)f(\eta^1)\nu_\rho^N(\eta).\end{aligned}$$
By the change of variables $\xi=\eta^1$, the previous expression can be written as
$$\sum_{\xi\in\Omega_N} f(\xi)\Big\{g(\xi^1)\xi(1)\alpha\frac{\nu_\rho^N(\xi^1)}{\nu_\rho^N(\xi)} + g(\xi^1)(1-\xi(1))(1-\alpha)\frac{\nu_\rho^N(\xi^1)}{\nu_\rho^N(\xi)}\Big\}\nu_\rho^N(\xi).$$
A simple computation shows that if $\xi(1)=1$, then $\frac{\nu_\rho^N(\xi^1)}{\nu_\rho^N(\xi)} = \frac{1-\rho}{\rho}$ so that the previous expression can be written as
$$\frac{\kappa}{N^\theta}\sum_{\xi\in\Omega_N} f(\xi)\Big\{g(\xi^1)\xi(1)\alpha\frac{1-\rho}{\rho} + g(\xi^1)(1-\xi(1))(1-\alpha)\frac{\rho}{1-\rho}\Big\}\nu_\rho^N(\xi),$$

from where we get, for $\alpha = \rho$, that

$$\int_{\Omega_N} g(\eta)c_1(\eta,\alpha)f(\eta^1)d\nu_\rho^N = \int_{\Omega_N} g(\eta^1)c_1(\eta^1,\rho)f(\eta)d\nu_\rho^N.$$

The same computation can be done if $\xi(1) = 0$, from where we conclude. We can repeat the same computation for the right boundary and this proves (2.4.2) for $\mathscr{L}_{N,b}$. This ends the proof of the lemma. □

When $\alpha \neq \beta$, the Bernoulli product measures are not reversible nor invariant. A simple way to check the non-invariance is to argue as follows. Suppose that the measures ν_ρ^N are invariant. Then we know that for any function $f : \Omega_N \to \mathbb{R}$ we have that

$$\int_{\Omega_N} \mathscr{L}_N f(\eta)d\nu_\rho^N = 0. \tag{2.4.4}$$

But for $f(\eta) = \eta(1)$, a simple computation shows that $\mathscr{L}_{N,0}f(\eta) = \frac{1}{2}(\eta(2) - \eta(1))$ and $\mathscr{L}^1_{N,b}f(\eta) = \frac{\kappa}{2N^\theta}[\alpha - \eta(1)]$, so that $\int_{\Omega_N} \mathscr{L}_N f(\eta)d\nu_\rho^N = \frac{\kappa}{2N^\theta}(\alpha - \rho)$ and this equals to 0 iff $\alpha = \rho$. Analogously, repeating the same computations as above for $f(\eta) = \eta(N-1)$, we would conclude (2.4.4) iff $\beta = \rho$. But this contradicts the fact that $\alpha \neq \beta$.

When $\alpha \neq \beta$, since we have a finite state irreducible Markov process, then there exists a unique stationary measure that we denote by μ_{ss}. A way to get information about this measure is to use the matrix ansatz method introduced in [6, 7]. The idea behind the method is the following. Let

$$f_{N-1}(\eta(1),\cdots,\eta(N-1))$$

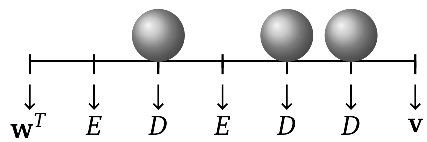
denote the weight of the configuration $\eta := (\eta(1),\cdots,\eta(N-1))$ with respect to the stationary measure μ_{ss} and let us suppose that

$$f_{N-1}(\eta(1),\eta(2),\cdots,\eta(N-1)) = \mathbf{w}^T X_{\eta(1)}X_{\eta(2)}\cdots X_{\eta(N-1)}\mathbf{v},$$

where

$$X_{\eta(x)} = \eta(x)D + (1-\eta(x))E,$$

and D, E are matrices (which in general do not commute) and the vectors $\mathbf{w}^T, \mathbf{v}$ are present in order to convert the matrix product into a scalar. In the figure below we take $N = 6$ and we present a possible configuration $\eta = (0,1,0,1,1)$ whose corresponding weight is given by $f_{N-1}(\eta) = \mathbf{w}^T EDEDD\mathbf{v}$.



Let $P(\eta(1), \eta(2), \cdots, \eta(N-1))$ be the normalized weight of the configuration $\eta := (\eta(1), \cdots, \eta(N-1))$ with respect to the stationary state μ_{ss}, which is given by

$$P(\eta(1), \eta(2), \cdots, \eta(N-1)) = \frac{f_{N-1}(\eta(1), \eta(2), \cdots, \eta(N-1))}{Z_{N-1}},$$

where Z_{N-1} is the sum of the weights of the 2^{N-1} possible configurations in Ω_N:

$$Z_{N-1} = \sum_{\eta(1)\in\{0,1\}} \cdots \sum_{\eta(N-1)\in\{0,1\}} f_{N-1}(\eta(1), \eta(2), \cdots, \eta(N-1)).$$

From the definition of f_{N-1}, we have that

$$P(\eta(1), \eta(2), \cdots, \eta(N-1)) = \frac{\mathbf{w}^T X_{\eta(1)} X_{\eta(2)} \cdots X_{\eta(N-1)} \mathbf{v}}{Z_{N-1}},$$

and the normalization can be written as

$$\begin{aligned} Z_{N-1} &= \sum_{\eta(1)\in\{1,0\}} \cdots \sum_{\eta(N-1)\in\{1,0\}} \mathbf{w}^T X_{\eta(1)} X_{\eta(2)} \cdots X_{\eta(N-1)} \mathbf{v} \\ &= \sum_{\eta(1)\in\{1,0\}} \cdots \sum_{\eta(N-2)\in\{1,0\}} \mathbf{w}^T X_{\eta(1)} X_{\eta(2)} \cdots X_{\eta(N-2)} (D+E) \mathbf{v} \\ &= \cdots = \mathbf{w}^T (D+E)^{N-1} \mathbf{v}. \end{aligned} \tag{2.4.5}$$

Let us now impose conditions on the matrices D and E. For that purpose, let $C = D+E$. The expectation of the occupation variable at the site x, with respect to the stationary state μ_{ss}, is given by

$$\begin{aligned} \rho_{ss}^N(x) = \int_{\Omega_N} \eta(x) d\mu_{ss} &= \frac{\sum_{\eta(1)\in\{1,0\}} \cdots \sum_{\eta(N-1)\in\{1,0\}} \eta(x) f_{N-1}(\eta(1), \cdots, \eta(N-1))}{Z_{N-1}} \\ &= \frac{1}{Z_{N-1}} \sum_{\eta(1)\in\{1,0\}} \cdots \sum_{\eta(N-1)\in\{1,0\}} \left[\mathbf{w}^T \prod_{j=1}^{x-1} X_{\eta(j)} D \prod_{j=x+1}^{N-1} X_{\eta(j)} \mathbf{v} \right] \\ &= \frac{\mathbf{w}^T C^{x-1} D C^{N-1-x} \mathbf{v}}{\mathbf{w}^T C^{N-1} \mathbf{v}}. \end{aligned} \tag{2.4.6}$$

Note that above the sum does not contain the factor $\eta(x) \in \{1,0\}$ since the expectation is non-zero iff $\eta(x) = 1$. We can also compute the expectation of the product of two point occupation variables at the sites x and y, with respect to the stationary state μ_{ss}, that is, for $1 \le x < y \le N-1$, we have that

$$\begin{aligned}\int_{\Omega_N} \eta(x)\eta(y)d\mu_{ss} =& \\ &= \frac{\sum_{\eta(1)\in\{0,1\}}\cdots\sum_{\eta(N-1)\in\{0,1\}} \eta(x)\eta(y) f_{N-1}(\eta(1),\cdots,\eta(N-1))}{Z_{N-1}} \\ &= \frac{\mathbf{w}^T C^{x-1} D C^{y-x-1} D C^{N-1-y}\mathbf{v}}{\mathbf{w}^T C^{N-1}\mathbf{v}}.\end{aligned}$$

Therefore, the two point correlation function, with respect to the stationary state μ_{ss}, is given on $1 \le x < y \le N-1$ by

$$\begin{aligned}\varphi_{ss}^N(x,y) :=& \int_{\Omega_N} (\eta(x)-\rho_{ss}^N(x))(\eta(y)-\rho_{ss}^N(y))d\mu_{ss} \\ =& \; \frac{\mathbf{w}^T C^{x-1} D C^{y-x-1} D C^{N-1-y}\mathbf{v}}{\mathbf{w}^T C^{N-1}\mathbf{v}} \\ -& \; \frac{\mathbf{w}^T C^{x-1} D C^{N-1-x}\mathbf{v}}{\mathbf{w}^T C^{N-1}\mathbf{v}} \cdot \frac{\mathbf{w}^T C^{y-1} D C^{N-1-y}\mathbf{v}}{\mathbf{w}^T C^{N-1}\mathbf{v}}.\end{aligned} \tag{2.4.7}$$

A simple computation (see [5]) shows that for the dynamics that we are considering in this chapter, the matrices D, E and the vectors $\mathbf{w}^T, \mathbf{v}$ satisfy the following relations:

$$\begin{aligned}& DE - ED = D + E = C, \\ & \mathbf{w}^T\left[\frac{\kappa\alpha}{2N^\theta}E - \frac{\kappa(1-\alpha)}{2N^\theta}D\right] = \mathbf{w}^T, \\ & \left[\frac{\kappa(1-\beta)}{2N^\theta}D - \frac{\kappa\beta}{2N^\theta}E\right]\mathbf{v} = \mathbf{v}.\end{aligned} \tag{2.4.8}$$

Moreover, we also have that

$$\begin{aligned}& \mathbf{w}^T D = \mathbf{w}^T[\alpha C - N^\theta], \\ & D\mathbf{v} = [N^\theta + \beta C]\mathbf{v}, \\ & \mathbf{w}^T E = \mathbf{w}^T\left[\frac{N^\theta}{\alpha} + \frac{1-\alpha}{\alpha}D\right].\end{aligned} \tag{2.4.9}$$

We note that the equations above also show that

$$C(D+I) = (D+E)(D+I) = DD + D + ED + E,$$

and that $C(D+I)=DD+DE=DC$. Analogously we have that $CD=(D-I)C$. Using (2.4.5), we obtain that Z_{N-1} for our dynamics is given by

$$Z_{N-1}=\frac{1}{(\alpha-\beta)^{N-1}}\frac{\Gamma(2N^\theta+N-1)}{\Gamma(2N^\theta)},$$

where $\Gamma(\cdot)$ denotes the Gamma function. For the details on these computations we refer the interested reader to [5]. Now, in (2.4.6), by writing $DC^{N-1-x}=DCC^{N-2-x}$ and using the fact that $C(D+I)=DC$ we obtain

$$\rho_{ss}^N(x)=\frac{\mathbf{w}^T C^{x-1}C(D+I)C^{N-2-x}\mathbf{v}}{Z_{N-1}}=\frac{\mathbf{w}^T C^{x}DC^{N-2-x}\mathbf{v}}{Z_{N-1}}+\frac{\mathbf{w}^T C^{N-2}\mathbf{v}}{Z_{N-1}}.$$

Repeating the procedure above and using the explicit expression for Z_{N-1} given above, we obtain a simple expression for $\rho_{ss}^N(x)$ given by

$$\rho_{ss}^N(x)=\beta+(N-x)\frac{\alpha-\beta}{2N^\theta+N-2}+(N^\theta-1)\frac{\alpha-\beta}{2N^\theta+N-2}. \tag{2.4.10}$$

In fact last identity can be rewritten as

$$\rho_{ss}^N(x)=\frac{\kappa(\beta-\alpha)x}{2N^\theta+N-2}+\alpha+\frac{\kappa(\beta-\alpha)x}{2N^\theta+N-2}\Big(\frac{N^\theta}{\kappa}-1\Big).$$

Analogously, from a simple, but long computation (see [5]), we have that

$$\int_{\Omega_N}\eta(x)\eta(y)d\mu_{ss}=\beta\rho_{ss}^N(x)+(N-y+N^\theta-1)\frac{\alpha-\beta}{2N^\theta+N-2}\rho_{ss}^{N-1}(x),$$

and from (2.4.10), we obtain

$$\begin{aligned}\int_{\Omega_N}\eta(x)\eta(y)d\mu_{ss}=\beta&\left[\frac{\beta(x+N^\theta-1)+\alpha(N-x+N^\theta-1)}{2N^\theta+N-2}\right]\\&+\frac{(N-y+N^\theta-1)(\alpha-\beta)}{2N^\theta+N-2}\left[\frac{\beta(x+N^\theta-1)+\alpha(N-x+N^\theta-2)}{2N^\theta+N-3}\right].\end{aligned}$$

Putting together last expresssions and doing simple, but long, computations we conclude that

$$\varphi_{ss}^N(x,y)=-\frac{(\alpha-\beta)^2(x+N^\theta-1)(N-y+N^\theta-1)}{(2N^\theta+N-2)^2(2N^\theta+N-3)}. \tag{2.4.11}$$

From the previous identity it follows that

$$\max_{x<y}|\varphi_{ss}^N(x,y)|=\begin{cases}O\Big(\frac{N^\theta}{N^2}\Big),\ \theta<1,\\ O\Big(\frac{1}{N^\theta}\Big),\ \theta\geq 1,\end{cases}\quad\rightarrow_{N\to\infty}0. \tag{2.4.12}$$

This means that as the size of the bulk tens to infinity, the two point correlation function vanishes. In the next subsection we analyse the empirical profile and the two point correlation function for more general initial measures.

2.5 Empirical profile and correlations

Before stating the hydrodynamic limit result we explain here how to have a guess on the form of the hydrodynamic equations by using the *empirical profile*. which was defined above in the case of the measure μ_{ss}. Now we generalize its definition. For a measure μ_N in Ω_N and for each $x \in \Lambda_N$ we denote by $\rho_t^N(x)$ the empirical profile at the site x, given by

$$\rho_t^N(x) \,=\, \mathbb{E}_{\mu_N}[\eta_{tN^2}(x)]\,.$$

We extend this definition to the boundary by setting

$$\rho_t^N(0) \,=\, \alpha \text{ and } \rho_t^N(N) \,=\, \beta\,, \text{ for all } t \geq 0\,.$$

Note that since μ_{ss} is a stationary measure the profile $\rho_{ss}^N(\cdot)$ does not depend on time, but now since μ_N is a general measure the empirical profile $\rho_t^N(\cdot)$ depends on time. From Kolmogorov's backward equation we know that $\rho_t^N(\cdot)$ is a solution of

$$\partial_t \rho_t^N(x) = \mathbb{E}_{\mu_N}[\mathscr{L}_N \eta_{tN^2}(x)].$$

A simple computation shows that

$$\mathscr{L}_N \eta(x) = j_{x-1,x}(\eta) - j_{x,x+1}(\eta)$$

where for $x \in \Lambda_N$, the quantity $j_{x,x+1}(\eta)$ denotes the microscopic current at the bond $\{x, x+1\}$, which is given by the difference between the jump rate from x to $x+1$ and the jump rate from $x+1$ to x. Note that for $x = 0$ (resp. $x = N-1$) $j_{x,x+1}$ is equal to the creation rate minus the annihilation rate at the site $x = 1$ (resp. $x = N-1$). Therefore

$$\begin{aligned}
j_{0,1}(\eta) &= \frac{\kappa}{2N^\theta}(\alpha - \eta(1)),\\
j_{x,x+1}(\eta) &= \frac{1}{2}(\eta(x) - \eta(x+1)),\\
j_{N-1,N}(\eta) &= \frac{\kappa}{2N^\theta}(\eta(N-1) - \beta).
\end{aligned} \tag{2.5.1}$$

A simple computation shows that $\rho_t^N(\cdot)$ is a solution of

$$\begin{cases} \partial_t \rho_t^N(x) \,=\, \big(N^2 \mathscr{B}_N^\theta \rho_t^N\big)(x)\,, \;\; x \in \Lambda_N\,, \;\; t \geq 0\,, \\ \rho_t^N(0) \,= \alpha\,, \rho_t^N(N) \,=\, \beta\,, \;\; t \geq 0\,, \end{cases} \tag{2.5.2}$$

where the operator $\mathscr{B}_N^\theta$ acts on functions $f:\Lambda_N\cup\{0,N\}\to\mathbb{R}$ as

$$\begin{cases} N^2(\mathscr{B}_N^\theta f)(x) \;=\; \frac{1}{2}\Delta_N f(x), \quad \text{for } x\in\{2,\cdots,N-2\}\,,\\ N^2(\mathscr{B}_N^\theta f)(1) = \frac{N^2}{2}(f(2)-f(1)) + \frac{\kappa N^2}{2N^\theta}(f(0)-f(1))\,,\\ N^2(\mathscr{B}_N^\theta f)(N-1) = \frac{N^2}{2}(f(N-2)-f(N-1)) + \frac{\kappa N^2}{2N^\theta}(f(N)-f(N-1)). \end{cases}$$

Above $\Delta_N f$ denotes the discrete Laplacian of f which is given on $x\in\Lambda_N$ by

$$\Delta_N f(x) = f(x+1)+f(x-1)-2f(x). \tag{2.5.3}$$

The stationary solution of (2.5.2) is given by

$$\rho_{ss}^N(x) = \mathbb{E}_{\mu_{ss}}[\eta_{tN^2}(x)] = a_N x + b_N$$

where

$$a_N = \frac{\kappa(\beta-\alpha)}{2N^\theta+\kappa(N-2)} \quad \text{and} \quad b_N = a_N\Big(\frac{N^\theta}{\kappa}-1\Big)+\alpha.$$

From this we get that

$$\lim_{N\to\infty}\max_{x\in\Lambda_N}\left|\rho_{ss}^N(x)-\bar\rho(\tfrac{x}{N})\right| = 0, \tag{2.5.4}$$

where for $q\in(0,1)$

$$\bar\rho(q) = \begin{cases} (\beta-\alpha)q+\alpha\,;\; \theta<1,\\ \frac{\kappa(\beta-\alpha)}{2+\kappa}q+\alpha+\frac{\beta-\alpha}{2+\kappa}\,;\; \theta=1,\\ \frac{\beta+\alpha}{2}\,;\; \theta>1. \end{cases} \tag{2.5.5}$$

Note that this will be a stationary solution of the hydrodynamic equation that we are looking for.

Now we obtain information about the two point correlation function. Let

$$V_N = \{(x,y)\in\{0,\cdots,N\}^2 : 0<x<y<N\},$$

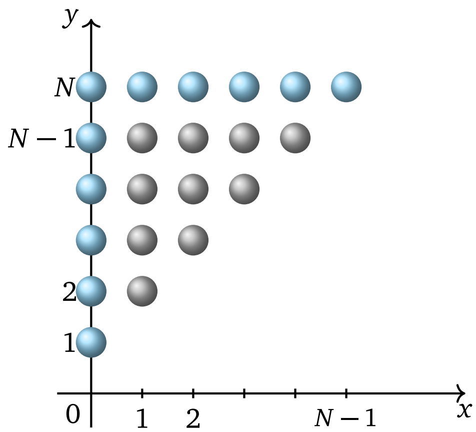
and its boundary $\partial V_N = \{(x,y)\in\{0,\cdots,N\}^2 : x=0 \text{ or } y=N\}$.

For $x<y\in V_N$, let $\varphi_t^N(x,y)$ denote the two point correlation function between the occupation sites at $x<y\in V_N$ which is defined by

$$\varphi_t^N(x,y) = \mathbb{E}_{\mu_N}[(\eta_{tN^2}(x)-\rho_t^N(x))(\eta_{tN^2}(y)-\rho_t^N(y))]. \tag{2.5.6}$$

Doing some simple, but long, computations we see that φ_t^N is a solution of

$$\begin{cases} \partial_t\varphi_t^N(x,y) = n^2\mathscr{A}_N^\theta\varphi_t^n(x,y)+g_t^N(x,y), & \text{for } (x,y)\in V_N,\ t>0,\\ \varphi_t^N(x,y)=0, & \text{for } (x,y)\in\partial V_N,\ t>0,\\ \varphi_0^N(x,y) = \mathbb{E}_{\mu_N}[\eta_0(x)\eta_0(y)]-\rho_0^N(x)\rho_0^N(y), & \text{for } (x,y)\in V_N\cup\partial V_N, \end{cases} \tag{2.5.7}$$



where $\mathscr{A}_N^\theta$ is the linear operator that acts on functions $f : V_N \cup \partial V_N \to \mathbb{R}$ as

$$(\mathscr{A}_N^\theta f)(u) = \sum_{v \in V_N} c_N^\theta(u,v)\big[f(v) - f(u)\big],$$

with

$$c_N^\theta(u,v) \;=\; \begin{cases} 1, & \text{if } \|u-v\| = 1 \text{ and } u,v \in V_N, \\ N^{-\theta}, & \text{if } \|u-v\| = 1 \text{ and } u \in V_N,\ v \in \partial V_N, \\ 0, & \text{otherwise,} \end{cases}$$

for $\theta \geq 0$. Note that $\mathscr{A}_N^\theta$ is the generator of a random walk in $V_N \cup \partial V_N$ with jump rates given by $c_N^\theta(u,v)$, which is absorbed at ∂V_N. Above $\|\cdot\|$ denotes the supremum norm,

$$g_t^N(x,y) = -(\nabla_N^+ \rho_t^N(x))^2 \delta_{y=x+1}$$

and

$$\nabla_N^+ \rho_t^N(x) = N(\rho_t^N(x+1) - \rho_t^N(x)).$$

In this case, contrarily to the empirical profile, is is quite complicated to obtain an expression for the stationary solution of (2.5.7). Nevertheless, we note that a simple, but long, computation shows that the solution obtained in (2.4.11), in the case where the starting measure is the stationary state μ_{ss}, is in fact the stationary solution of (2.5.7). We also observe that in [10] it was obtained the following bound on the case $\theta = 1$ for a general initial measure μ_N. There it was proved that there exists a constant $C > 0$ such that

$$\sup_{t \geq 0} \max_{(x,y) \in V_N} |\varphi_t^N(x,y)| \;\leq\; \frac{C}{N}, \tag{2.5.8}$$

but we note that the bounds on the other regimes of θ are still open, apart the case $\theta = 0$ where the bound above is given by C/N^2, see [17].

2.6 Hydrodynamic equations

From now on up to the rest of these notes we fix a finite time horizon $[0, T]$. We denote by $\langle \cdot, \cdot \rangle_\mu$ the inner product in $L^2([0,1])$ with respect to a measure μ defined in $[0,1]$ and $\| \cdot \|_{L^2(\mu)}$ is the corresponding norm. We note that when μ is the Lebesgue measure we write $\langle \cdot, \cdot \rangle$ and $\| \cdot \|_{L^2}$ for the corresponding norm.

We denote by $C^{m,n}([0,T] \times [0,1])$ the set of functions defined on $[0,T] \times [0,1]$ that are m times differentiable on the first variable and n times differentiable on the second variable. For a function $G := G(s,q) \in C^{m,n}([0,T] \times [0,1])$ we denote by $\partial_s G$ its derivative with respect to the time variable s and and by $\partial_q G$ its derivative with respect to the space variable q. For simplicity of notation we set $\Delta G := \partial_q^2 G$. We will also make use of the set $C_c^{m,n}([0,T] \times [0,1])$ of functions $G \in C^{m,n}([0,T] \times [0,1])$ such that for any time s the function G_s has a compact support included in $(0,1)$ and we denote by $C_c^m(0,1)$ (resp. $C_c^\infty(0,1)$) the set of all m continuously differentiable (resp. smooth) real-valued functions defined on $(0,1)$ with compact support. The supremum norm is denoted by $\| \cdot \|_\infty$. Finally, $C_0^{m,n}([0,T] \times [0,1])$ is the set of functions $G \in C^{m,n}([0,T] \times [0,1])$ such that for any time s the function G_s vanishes at the boundary, that is, $G_s(0) = G_s(1) = 0$.

Now we want to define the space where the solutions of the hydrodynamic equations will live on, namely the Sobolev space $\mathscr{H}_1$ on $[0,1]$. For that purpose, we define the semi inner-product $\langle \cdot, \cdot \rangle_1$ on the set $C^\infty([0,1])$ by

$$\langle G, H \rangle_1 = \int_0^1 (\partial_q G)(q) \, (\partial_q H)(q) \, dq, \tag{2.6.1}$$

for $G, H \in C^\infty([0,1])$ and the corresponding semi-norm is denoted by $\| \cdot \|_1$.

Definition 2.6.1. *The Sobolev space $\mathscr{H}^1$ on $[0,1]$ is the Hilbert space defined as the completion of $C^\infty([0,1])$ for the norm*

$$\| \cdot \|^2_{\mathscr{H}^1} := \| \cdot \|^2_{L^2} + \| \cdot \|^2_1.$$

Its elements elements coincide a.e. with continuous functions. The space $L^2(0,T; \mathscr{H}^1)$ is the set of measurable functions $f : [0,T] \to \mathscr{H}^1$ such that

$$\int_0^T \|f_s\|^2_{\mathscr{H}^1} ds < \infty.$$

We can now give the definition of the weak solutions of the hydrodynamic equations that will be derived for the symmetric simple exclusion process in contact with stochastic reservoirs. We start by giving the notion of a weak solution to the heat equation with Dirichlet boundary conditions which will be the notion that we will derive in the regime $\theta \in [0,1)$. In what follows $g : [0,1] \to [0,1]$ is a measurable function and it is the initial condition of all the partial differential equations that we define below, that is $\rho_0(q) = g(q)$, for all $q \in (0,1)$.

Definition 2.6.2. *We say that $\rho : [0,T] \times [0,1] \to [0,1]$ is a weak solution of the heat equation with Dirichlet boundary conditions*

$$\begin{cases} \partial_t \rho_t(q) = \frac{1}{2}\Delta \rho_t(q), & (t,q) \in [0,T] \times (0,1), \\ \rho_t(0) = \alpha, \quad \rho_t(1) = \beta, & t \in [0,T], \end{cases} \tag{2.6.2}$$

if the following two conditions hold:

1. $\rho \in L^2(0,T;\mathscr{H}^1)$;
2. ρ *satisfies the weak formulation:*

$$\begin{aligned} F_{Dir} := \int_0^1 \rho_t(q) G_t(q)\, dq - \int_0^1 g(q) G_0(q)\, dq \\ - \int_0^t \int_0^1 \rho_s(q) \Big(\frac{1}{2}\Delta + \partial_s \Big) G_s(q)\, dq\, ds \\ + \int_0^t \Big\{ \frac{\beta}{2} \partial_q G_s(1) - \frac{\alpha}{2} \partial_q G_s(0) \Big\}\, ds = 0, \end{aligned} \tag{2.6.3}$$

for all $t \in [0,T]$ and any function $G \in C_0^{1,2}([0,T] \times [0,1])$.

In the regime $\theta < 0$ we will make use of another notion of weak solution to the heat equation with Dirichlet boundary conditions which uses as input for test functions elements in the set $C_c^{1,2}([0,T] \times [0,1])$. Since functions in that space have compact support, in order to get a proper notion of weak solution we need to add an extra condition to Definition 2.6.2 (see 3. in Definition 2.6.3).

Definition 2.6.3. *We say that $\rho : [0,T] \times [0,1] \to [0,1]$ is a weak solution of the heat equation with Dirichlet boundary conditions given in* (2.6.2) *if the following three conditions hold:*

1. $\rho \in L^2(0,T;\mathscr{H}^1)$,

2. *ρ satisfies the weak formulation:*

$$
\begin{aligned}
F_{Dir}^{c} := \int_0^1 \rho_t(q) G_t(q)\, dq - \int_0^1 g(q) G_0(q)\, dq \\
- \int_0^t \int_0^1 \rho_s(q) \Big(\frac{1}{2}\Delta + \partial_s \Big) G_s(q)\, dq\, ds = 0,
\end{aligned}
\tag{2.6.4}
$$

for all $t \in [0,T]$ and any function $G \in C_c^{1,2}([0,T]\times[0,1])$,

3. $\rho_t(0) = \alpha, \quad \rho_t(1) = \beta$ *for all $t \in (0,T]$.*

Remark 2.6.4. *We note that* (2.6.4) *coincides with* (2.6.3) *by taking as input a test function $G \in C_c^{1,2}([0,T]\times[0,1])$, since in this case $\partial_q G_s(0) = \partial_s G_s(1) = 0$, so that the last term in* (2.6.3) *vanishes.*

Now we introduce the notion of weak solution of the hydrodynamic equation that we will derive in the case $\theta = 1$. In this regime the boundary reservoirs are so slow and as a consequence, a different boundary condition appears. In the case of Dirichlet boundary conditions, the value of the profile ρ_t is fixed to be equal to α at 0 and β at 1. This is no longer the case when $\theta \geq 1$ as we will see later on.

Definition 2.6.5. *We say that $\rho : [0,T]\times[0,1] \to [0,1]$ is a weak solution of the heat equation with Robin boundary conditions*

$$
\begin{cases}
\partial_t \rho_t(q) = \frac{1}{2}\Delta \rho_t(q), \quad (t,q) \in [0,T]\times(0,1), \\
\partial_q \rho_t(0) = \kappa(\rho_t(0) - \alpha), \quad \partial_q \rho_t(1) = \kappa(\beta - \rho_t(1)), \quad t \in [0,T],
\end{cases}
\tag{2.6.5}
$$

if the following two conditions hold:

1. $\rho \in L^2(0,T;\mathscr{H}^1)$,

2. *ρ satisfies the weak formulation:*

$$
\begin{aligned}
F_{Rob} &:= \int_0^1 \rho_t(q) G_t(q)\, dq - \int_0^1 g(q) G_0(q)\, dq \\
&- \int_0^t \int_0^1 \rho_s(q) \Big(\frac{1}{2}\Delta + \partial_s \Big) G_s(q)\, ds\, dq + \frac{1}{2}\int_0^t \{\rho_s(1)\partial_q G_s(1) - \rho_s(0)\partial_q G_s(0)\}\, ds \\
&- \frac{\kappa}{2}\int_0^t \{G_s(0)(\alpha - \rho_s(0)) + G_s(1)(\beta - \rho_s(1))\}\, ds = 0,
\end{aligned}
\tag{2.6.6}
$$

for all $t \in [0,T]$ and any function $G \in C^{1,2}([0,T]\times[0,1])$.

In the regime $\theta = 1$ the boundary reservoirs are so slow so that a type of Robin boundary condition appears. In this case it fixes the value of the flux through the system as being proportional to the difference of concentration. Note that, for example at $q = 0$, the value $\partial_q \rho_t(0)$ corresponds to the flux of particles through the left boundary and $\kappa(\rho_t(0) - \alpha)$ corresponds to the difference of the concentration, since in this case, contrarily to what happens in the case of Dirichlet boundary conditions, it is not true that $\rho_t(0) = \alpha$ (the value of the profile at the boundaries is not fixed!)

Remark 2.6.6. *Observe that in the case $\kappa = 0$ the equation above is the heat equation with Neumann boundary conditions and it is the hydrodynamic equation that we will derive in the case $\theta > 1$.*

Remark 2.6.7. *We observe that all the partial differential equations defined above have a unique weak solution in the sense given above. We do not include the proof of this result in these notes but we refer the interested reader to [2] for the proof of the uniqueness in the case of Dirichlet boundary conditions and to [1] for the proof of the uniqueness in the case of Robin boundary conditions.*

- Deriving the weak formulation

We note that the weak formulation given in all the regimes above can be obtained from the formal expression of the corresponding partial differential equation in the following way. Take a test function $G \in C^{1,2}([0,T] \times [0,1])$ and multiply both sides of the equality

$$\partial_s \rho_s(q) = \frac{1}{2} \Delta \rho_s(q)$$

by G and then integrate both in time and space to get

$$\int_0^1 \int_0^t \partial_s \rho_s(q) G_s(q) \, ds \, dq = \int_0^1 \int_0^t \frac{1}{2} \Delta \rho_s(q) G_s(q) \, ds \, dq. \tag{2.6.7}$$

To treat the term at the left hand side of last display, we perform an integration by parts in the time integral and we get to

$$\int_0^1 \rho_t(q) G_t(q) \, dq - \int_0^1 g(q) G_0(q) \, dq - \int_0^t \int_0^1 \rho_s(q) \partial_s G_s(q) \, ds \, dq. \tag{2.6.8}$$

The term at the right hand side of (2.6.7) can be treated by doing an integration by parts in the space integral and we get to

$$\frac{1}{2} \int_0^t \partial_q \rho_s(1) G_s(1) - \partial_q \rho_s(0) G_s(0) \, ds - \frac{1}{2} \int_0^t \int_0^1 \partial_q \rho_s(q) \partial_q G_s(q) \, ds \, dq.$$

Now, we do another integration by parts in the integral in space at the term on the right hand side of last expression and we write the previous display as

$$\begin{aligned}&\frac{1}{2}\int_0^t \partial_q\rho_s(1)G_s(1)-\partial_q\rho_s(0)G_s(0)\,ds\\ &-\frac{1}{2}\int_0^t \rho_s(1)\partial_q G_s(1)-\rho_s(0)\partial_q G_s(0)+\frac{1}{2}\int_0^t\int_0^1 \rho_s(q)\Delta G_s(q)\,ds\,dq.\end{aligned}\tag{2.6.9}$$

Putting together (2.6.9) and (2.6.8) we obtain

$$\begin{aligned}\int_0^1 \rho_t(q)G_t(q)\,dq-\int_0^1 g(q)G_0(q)\,dq=&\int_0^t\int_0^1 \rho_s(q)\Big(\frac{1}{2}\Delta+\partial_s\Big)G_s(q)\,ds\,dq\\ &+\frac{1}{2}\int_0^t \partial_q\rho_s(1)G_s(1)-\partial_q\rho_s(0)G_s(0)\,ds\\ &-\frac{1}{2}\int_0^t \rho_s(1)\partial_q G_s(1)-\rho_s(0)\partial_q G_s(0)\,ds.\end{aligned}$$

Now we obtain each one of the weak formulations given above. We start with the case where $G\in C_0^{1,2}([0,T]\times[0,1])$ and we will derive (2.6.3). For that purpose note that since G vanishes at the boundary of $[0,1]$ and since $\rho_s(0)=\alpha$ and $\rho_s(1)=\beta$, the expression in the previous display becomes equivalent to $F_{Dir}=0$. On the other hand if $G\in C_c^{1,2}([0,T]\times[0,1])$, then G vanishes at the boundary of $[0,1]$ and $\partial_q G$ also vanishes at the boundary of $[0,1]$, so that for ρ satisfying the Dirichlet boundary conditions in (2.6.2) the expression in the display above becomes equivalent to $F^c_{Dir}=0$. Finally for $G\in C^{1,2}([0,T]\times[0,1])$ and for ρ satisfying the Robin boundary conditions in (2.6.5), the expression in the previous display becomes equivalent to $F_{Rob}=0$.

-Stationary solutions

Now we deduce the stationary solutions for each one of the equations given above. We start with (2.6.2). For that purpose note that, denoting by $\bar\rho$ the stationary solution we have that $\Delta\bar\rho(t,q)=0$ implies that $\bar\rho(t,q)=aq+b$ for $a,b\in\mathbb{R}$. Imposing the Dirichlet boundary conditions we arrive at $a=(\beta-\alpha)$ and $b=\beta$, so that

$$\bar\rho_{Dir}(q)=(\beta-\alpha)q+\alpha.\tag{2.6.10}$$

On the other hand, imposing the Robin boundary conditions we arrive at

$$a=\frac{\kappa(\beta-\alpha)}{2+\kappa}\quad\text{and}\quad b=\alpha+\frac{\beta-\alpha}{2+\kappa},$$

so that

$$\bar{\rho}_{Rob}(q) = \frac{\kappa(\beta-\alpha)}{2+\kappa}q + \alpha + \frac{\beta-\alpha}{2+\kappa}. \tag{2.6.11}$$

Finally, if we impose the Neumann boundary conditions, any constant solution is a stationary solution of (2.6.5) with $\kappa = 0$ (which corresponds to the Neumann regime). In this case we note that the stationary solution is not unique. Below we draw the graph of these stationary solutions for a choice of $\alpha = 0.2$ and $\beta = 0.8$.

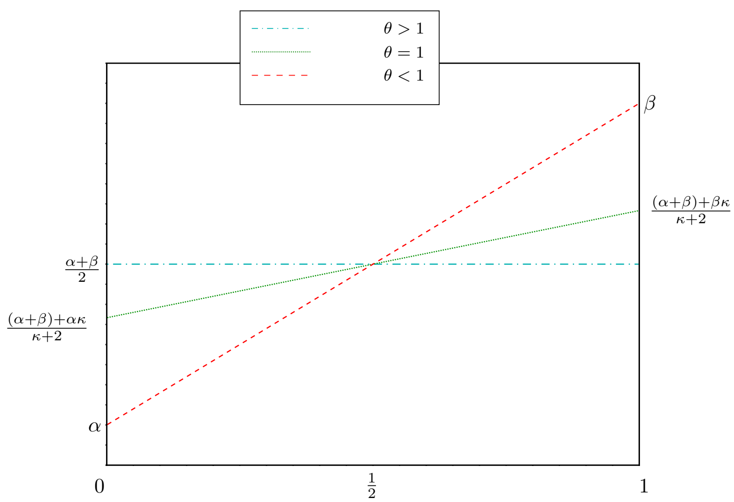


Figure 2.3: Stationary solutions of the hydrodynamic equations.

Now we give the explicit expression for the solution of each hydrodynamic equation.

Proposition 2.6.8. *We have that:*

1. *The solution of* (2.6.2) *with initial condition* g *is equal to*

$$\rho_t(q) = \bar{\rho}_{Dir}(q) + \sum_{n=1}^{\infty} e^{-\frac{(n\pi)^2}{2}t} 2\sin(n\pi q).$$

2. *The solution of* (2.6.5) *with initial condition* g *is equal to*

$$\rho_t(q) = \bar{\rho}_{Rob}(q) + \sum_{n=1}^{\infty} C_n e^{-\frac{\lambda_n}{2}t} X_n(q),$$

where

$$X_n(q) = A_n \sin(\sqrt{\lambda_n}\, q) + A_n \kappa \sqrt{\lambda_n} \cos(\sqrt{\lambda_n}\, q), \tag{2.6.12}$$

A_n is a normalizing constant in such a way that X_n has unitary $L^2([0,1])$-norm and

$$C_n = \int_0^1 (g(q) - \bar{\rho}(q)) X_n(q) dq.$$

Proof. The solution ρ to (2.6.2) starting from a profile g is such that $u = \rho - \bar{\rho}$ is solution to (2.6.2) with homogeneous boundary conditions $\alpha = \beta = 0$, i.e.

$$\begin{cases} \partial_t u_t(q) = \frac{1}{2}\Delta u_t(q), & (t,q) \in [0,T] \times (0,1), \\ u_t(0) = 0 = u_t(1), & t \in [0,T]. \end{cases} \tag{2.6.13}$$

It is well known that u is given by

$$u_t(q) = \sum_{n=1}^{\infty} e^{-\frac{(n\pi)^2}{2}t} 2\sin(n\pi q).$$

From the previous computations we conclude that the solution ρ of (2.6.2) starting from g is given by

$$\rho_t(q) = (\beta - \alpha)q + \alpha + \sum_{n=1}^{\infty} e^{-\frac{(n\pi)^2}{2}t} 2\sin(n\pi q).$$

On the other hand, the solution ρ of (2.6.5) starting from g is such that $u = \rho - \bar{\rho}$ is solution to (2.6.5) with $\alpha = \beta = 0$, i.e.

$$\begin{cases} \partial_t u_t(q) = \frac{1}{2}\Delta u_t(q), & (t,q) \in [0,T] \times (0,1), \\ \partial_q u_t(0) = \kappa u_t(0), \quad \partial_q u_t(1) = -\kappa u_t(1), & t \in [0,T]. \end{cases} \tag{2.6.14}$$

It is well known that u is given by

$$u(t,q) = \sum_{n=1}^{\infty} C_n e^{-\frac{\lambda_n}{2}t} X_n(q),$$

where $X_n(q)$ writes as

$$X_n(q) = A_n \sin(\sqrt{\lambda_n} q) + B_n \cos(\sqrt{\lambda_n} q),$$

for some constants A_n and B_n. Then, the first boundary condition in (2.6.14) gives $B_n = \sqrt{\lambda_n}\kappa A_n$. To avoid the null solution we consider $A_n \neq 0$. The second boundary condition in (2.6.14) gives

$$\tan(\sqrt{\lambda_n}) = \frac{2\kappa\sqrt{\lambda_n}}{\lambda_n \kappa^2 - 1}, \tag{2.6.15}$$

whose solution λ_n satisfying $(n-1)\pi \leq \sqrt{\lambda_n} \leq n\pi$ is such that $\lambda_n \sim n^2\pi^2$ as $n \to \infty$. From the previous computations we get that $X_n(q)$ is given by (2.6.12) and there A_n is a normalizing constant in such a way that X_n has unitary $L^2([0,1])$-norm. Moreover

$$C_n = \int_0^1 (g(q) - \bar{\rho}(q))X_n(q)dq.$$

From the previous computations we conclude that the solution ρ of (2.6.5) starting from g is given by

$$\rho_t(q) = \frac{\kappa(\beta-\alpha)}{2+\kappa}q + \alpha + \frac{\beta-\alpha}{2+\kappa} + \sum_{n=1}^{\infty} C_n e^{-\frac{\lambda_n}{2}t} X_n(q).$$

$\square$

2.7 Hydrodynamic Limit

In this section we want to state the hydrodynamic limit of the process $\{\eta_{tN^2} : t \geq 0\}$ with state space Ω_N and with infinitesimal generator $N^2\mathscr{L}_N$ defined in (2.3.1). Note that here we are going to take $\Theta(N) = N^2$. Let $\mathscr{M}^+$ be the space of positive measures on $[0,1]$ with total mass bounded by 1 equipped with the weak topology. For any configuration $\eta \in \Omega_N$ we define the empirical measure $\pi^N(\eta, dq)$ on $[0,1]$ by

$$\pi^N(\eta, dq) = \frac{1}{N-1} \sum_{x \in \Lambda_N} \eta(x)\delta_{\frac{x}{N}}(dq), \tag{2.7.1}$$

where δ_a is a Dirac mass on $a \in [0,1]$, and

$$\pi_t^N(\eta, dq) := \pi^N(\eta_{tN^2}, dq).$$

This measure gives weight $\frac{1}{N}$ to each occupied site of the configuration η.

Fix $T > 0$ and $\theta \in \mathbb{R}$. Recall that $\mathbb{P}_{\mu_N}$ is the probability measure in the Skorohod space $\mathscr{D}([0,T],\Omega_N)$ induced by the Markov process $\{\eta_{tN^2} : t \geq 0\}$ and the initial probability measure μ_N and we denote by $\mathbb{E}_{\mu_N}$ the expectation with respect to $\mathbb{P}_{\mu_N}$. Now let $\{\mathbb{Q}_N\}_{N\geq 1}$ be the sequence of probability measures on $\mathscr{D}([0,T],\mathscr{M}^+)$ induced by the Markov process $\{\pi_t^N : t \geq 0\}$ and by $\mathbb{P}_{\mu_N}$.

At this point we need to fix an initial profile $\rho_0 : [0,1] \to [0,1]$ which is measurable and an initial probability measure $\mu_N \in \Omega_N$. We are going to consider the following set of initial measures:

Definition 2.7.1. *A sequence of probability measures $\{\mu_N\}_{N\geq 1}$ in Ω_N is associated to the profile ρ_0 if for any continuous function $G:[0,1]\to\mathbb{R}$ and any $\delta>0$*

$$\lim_{N\to\infty}\mu_N\Big(\eta\in\Omega_N:\Big|\frac{1}{N-1}\sum_{x\in\Lambda_N}G\left(\tfrac{x}{N}\right)\eta(x)-\int_0^1 G(q)\rho_0(q)dq\Big|>\delta\Big)=0. \tag{2.7.2}$$

Note that (2.7.2) states that

$$\int_{\Omega_N} G(q)\pi^N(\eta,dq)\longrightarrow_{N\to\infty}\int_0^1 G(q)\rho_0(q)dq \quad \text{wrt } \mu_N, \tag{2.7.3}$$

which means that the empirical measure at time $t=0$ converges, in probability with respect to μ_N, as $N\to\infty$, to the deterministic measure $\rho_0(q)dq$, which is absolutely continuous with respect to the Lebesgue measure and the density is the profile $\rho_0(\cdot)$.

The hydrodynamic limit that we want to derive states that the previous result is also true for any $t\in[0,T]$, that is, the empirical measure at time t converges in probability with respect to the distribution of the system at time t, as $N\to\infty$, to the deterministic measure $\rho_t(q)dq$, where $\rho_t(\cdot)$ is a solution (here in the weak sense) to some partial differential equation, *the hydrodynamic equation*.

The first main result of these notes is summarized in the following theorem (see also Figure 2.4).

Theorem 2.7.2. *Let $g:[0,1]\to[0,1]$ be a measurable function and let $\{\mu_N\}_{N\geq 1}$ be a sequence of probability measures in Ω_N associated to g. Then, for any $t\in[0,T]$,*

$$\lim_{N\to\infty}\mathbb{P}_{\mu_N}\Big(\eta_\cdot:\Big|\frac{1}{N-1}\sum_{x\in\Lambda_N}G\left(\tfrac{x}{N}\right)\eta_{tN^2}(x)-\int_0^1 G(q)\rho_t(q)dq\Big|>\delta\Big)=0,$$

where $\rho_t(\cdot)$ is the unique weak solution of :

- (2.6.3) *as given in Definition 2.6.3, if $\theta<0$;*
- (2.6.2) *as given in Definition 2.6.2, if $\theta\in[0,1)$;*
- *(2.6.5), if $\theta=1$;*
- *(2.6.5) with $\kappa=0$, if $\theta>1$.*

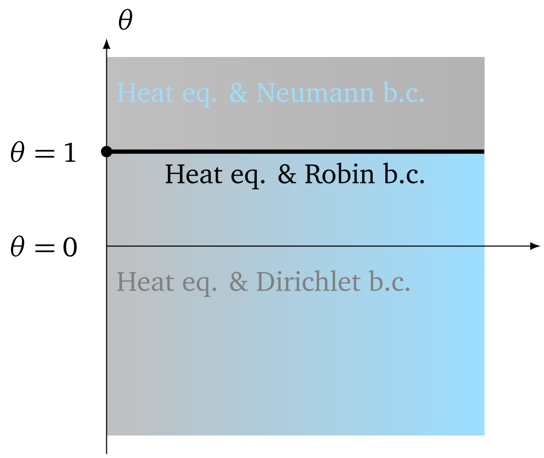


Figure 2.4: The three hydrodynamic equations depending on θ.

The proof of Theorem 2.7.2 proceeds as follows. We split the proof into showing first tightness of the sequence $\{\mathbb{Q}_N\}_{N\geq 1}$ and then we characterize uniquely the limiting point $\mathbb{Q}$. These two results combined together, imply the convergence of $\{\mathbb{Q}_N\}_{N\geq 1}$ to $\mathbb{Q}$ as $N \to \infty$. The next section is dedicated to the presentation of an heuristic argument to deduce the hydrodynamic equations from the interacting particle system by means of the Dynkin's formula; in Section 2.9 we present the proof of tightness and in Section 2.10 we characterize the limit point $\mathbb{Q}$. We note that in order to characterize the limit point $\mathbb{Q}$, we prove in Section 2.10.1 that all limiting points of the sequence $\{\mathbb{Q}_N\}_{N\geq 1}$ are concentrated on trajectories of measures that are absolutely continuous with respect to the Lebesgue measure and in Sections 2.10.2 and 2.10.3 we prove that the density $\rho_t(\cdot)$ is a weak solution of the corresponding hydrodynamic equation. From the uniqueness of weak solutions of the hydrodynamic equations, see Remark 2.6.7, we conclude that $\{\mathbb{Q}_N\}_{N\geq 1}$ has a unique limit point $\mathbb{Q}$, and therefore we conclude the convergence of the sequence to this limit point.

2.8 Heuristics for hydrodynamic equations

In this section we give the main ideas which are behind the identification of limit points as weak solutions of the partial differential equations given in Section 2.6. Now we argue that the density $\rho_t(\cdot)$ is a weak solution of the corresponding hydrodynamic equation for each regime of θ. We remark that we are not going to prove here that the solution $\rho_t(\cdot)$ belongs to the space $L^2(0, T; \mathscr{H}^1)$ but we refer the reader to [1, 2] for a complete proof of this fact. In order to prove that $\rho_t(\cdot)$ satisfies the weak formulation we use auxiliary martingales associated to the Markov process $\{\eta_t : t \geq 0\}$. For that purpose, and to make the exposition simpler, we fix a function $G : [0,1] \to \mathbb{R}$ which does not depend on time and which is two times continuously differentiable. If $\theta < 0$ we will assume further that it has a compact support included in $(0,1)$. First we recall Dynkin's formula.

Theorem 2.8.1. *Let $\{\eta_t : t \geq 0\}$ be a Markov process with generator $\mathscr{L}$ and with countable state space E. Let $F : \mathbb{R}^+ \times E \to \mathbb{R}$ be a bounded function such that*

- $\forall \eta \in E, F(\cdot, \eta) \in C^2(\mathbb{R}^+)$,
- *there exists a finite constant C, such that for $j = 1, 2$*

$$\sup_{(s,\eta)} |\partial_s^j F(s,\eta)| \leq C.$$

For $t \geq 0$, let

$$\begin{aligned} M_t^F =& F(t,\eta_t) - F(0,\eta_0) - \int_0^t (\partial_s + \mathscr{L}) F(s,\eta_s) ds, \\ N_t^F =& (M_t^F)^2 - \int_0^t \{\mathscr{L} F(s,\eta_s)^2 - 2F(s,\eta_s)\mathscr{L} F(s,\eta_s)\} ds. \end{aligned}$$

Then, $\{M_t^F\}_{t\geq 0}$ and $\{N_t^F\}_{t\geq 0}$ are martingales with respect to $\mathscr{F}_s = \sigma(\eta_s\,; s \leq t)$.

Let us fix a test function $G : [0,1] \to \mathbb{R}$ and apply Dynkin's formula with

$$F(t,\eta_t) = \langle \pi_t^N, G\rangle = \frac{1}{N-1} \sum_{x\in\Lambda_N} \eta_{tN^2}(x) G\Big(\tfrac{x}{N}\Big). \tag{2.8.1}$$

Above $\big\langle \pi_t^N, G \big\rangle$ represents the integral of G with respect the measure π_t^N. Note that F does not depend on time, only through η_t. A simple computation shows that

$$\begin{aligned} N^2 \mathscr{L}_N \langle \pi_s^N, G\rangle = & \langle \pi_s^N, \frac{1}{2}\Delta_N G\rangle \\ & + \frac{1}{2}\Big(\nabla_N^+ G(0)\eta_{sN^2}(1) - \nabla_N^- G(1)\eta_{sN^2}(N-1)\Big) \\ & + \frac{\kappa}{2}\frac{N^{2-\theta}}{N-1} G\Big(\tfrac{1}{N}\Big)(\alpha - \eta_{sN^2}(1)) \\ & + \frac{\kappa}{2}\frac{N^{2-\theta}}{N-1} G\Big(\tfrac{N-1}{N}\Big)(\beta - \eta_{sN^2}(N-1)), \end{aligned} \tag{2.8.2}$$

from where we obtain that

$$\begin{aligned} M_t^N(G) = \langle \pi_t^N, G\rangle - \langle \pi_0^N, G\rangle - \int_0^t \langle \pi_s^N, \frac{1}{2}\Delta_N G\rangle\, ds \\ - \frac{1}{2}\int_0^t \nabla_N^+ G(0)\eta_{sN^2}(1) - \nabla_N^- G(1)\eta_{sN^2}(N-1)\, ds \\ - \frac{\kappa}{2}\int_0^t \frac{N^{2-\theta}}{N-1} G\Big(\tfrac{1}{N}\Big)(\alpha - \eta_{sN^2}(1))\, ds \\ - \frac{\kappa}{2}\int_0^t \frac{N^{2-\theta}}{N-1} G\Big(\tfrac{N-1}{N}\Big)(\beta - \eta_{sN^2}(N-1))\, ds, \end{aligned} \tag{2.8.3}$$

is a martingale with respect to the natural filtration $\{\mathscr{F}_t\}_{t\geq 0}$, where for each $t \geq 0$, $\mathscr{F}_t := \sigma(\eta_s : s < t)$. Now we look at the integral terms in (2.8.3).

- The case $\theta \in [0,1)$:

In this regime, we take a test function $G : [0,1] \to \mathbb{R}$ two times continuously differentiable such that $G(0) = G(1) = 0$. Then, we can subtract $G(0)$ (resp. $G(1)$) in the fifth term (resp. sixth term) at the right hand side of (2.8.3) and then doing a Taylor expansion on G we get that

$$\begin{aligned} M_t^N(G) = \langle \pi_t^N, G\rangle - \langle \pi_0^N, G\rangle - \int_0^t \langle \pi_s^N, \frac{1}{2}\Delta_N G\rangle ds \\ - \frac{1}{2}\int_0^t \nabla_N^+ G(0)\eta_{sN^2}(1) - \nabla_N^- G(1)\eta_{sN^2}(N-1) ds + O(N^{-\theta}). \end{aligned}$$

If we can replace $\eta_{sN^2}(1)$ by α and $\eta_{sN^2}(N-1)$ by β, which will be a consequence of Lemma A.4.2 (see Remark A.4.3), then above we have

$$\begin{aligned} M_t^N(G) = \langle \pi_t^N, G\rangle - \langle \pi_0^N, G\rangle - \int_0^t \langle \pi_s^N, \frac{1}{2}\Delta_N G\rangle ds \\ - \frac{1}{2}\int_0^t \nabla_N^+ G(0)\alpha - \nabla_N^- G(1)\beta\, ds + O(N^{-\theta}) \end{aligned}$$

plus a term that vanishes as $N \to +\infty$.

Taking the expectation with respect to μ_N in the expression above we get

$$\frac{1}{N-1}\sum_{x=1}^{N-1} G\Big(\tfrac{x}{N}\Big)\Big(\rho_t^N(x)-\rho_0^N(x)\Big)-\int_0^t \frac{1}{N-1}\sum_{x=1}^{N-1}\frac{1}{2}\Delta_N G\Big(\tfrac{x}{N}\Big)\rho_s^N(x)ds$$
$$-\frac{1}{2}\int_0^t \nabla_N^+ G(0)\alpha-\nabla_N^- G(1)\beta\, ds+O(N^{-\theta})=0.$$

Note that above we used the fact that the average of martingales is constant in time and that $M_0^N(G)=0$. Now, assuming that $\rho_t^N(x)\sim\rho_t(\frac{x}{N})$ and taking the limit as $N\to\infty$ we get that

$$\int_0^1 \rho_t(q)G(q)-\rho_0(q)G(q)dq-\int_0^t\int_0^1 \frac{1}{2}\Delta G(q)\rho_s(q)dqds$$
$$-\frac{1}{2}\int_0^t \partial_q G(0)\alpha-\partial_q G(1)\beta\, ds=0.$$

Note that the restriction $\theta\geq 0$ comes from the fact that the errors, which arise from the Taylor expansion in G, have to vanish as $N\to\infty$ and the restriction $\theta<1$ comes from the replacement of the occupation variables $\eta(1)$ and $\eta(N-1)$ by α and β, respectively, see Lemma A.4.2. At this point compare the previous expression with the weak formulation given in (2.6.3) and note that the test function G does not depend on time.

- The case $\theta<0$:

In this regime we take a function $G:[0,1]\to\mathbb{R}$ with compact support and we note that the last three terms at the right hand side of (2.8.3) vanish in this case. From this and the same arguments as above we get that

$$M_t^N(G)=\langle\pi_t^N,G\rangle-\langle\pi_0^N,G\rangle-\int_0^t\langle\pi_s^N,\frac{1}{2}\Delta_N G\rangle ds.$$

Taking the expectation with respect to μ_N in the expression above and assuming that $\rho_t^N(x)\sim\rho_t(\frac{x}{N})$, and then taking the limit as $N\to\infty$ we get that

$$\int_0^1 \rho_t(q)G(q)-\rho_0(q)G(q)dq-\int_0^t\int_0^1\frac{1}{2}\Delta G(q)\rho_s(q)dqds=0.$$

Again compare with the weak formulation given in (2.6.4) and note that the test function G does not depend on time.

Remark 2.8.2. *We remark here that in this particular case there is an extra condition in Definition 2.6.3 with respect to the other notions of weak solutions where we only have to check the weak formulation and to show that the solution belongs to a Sobolev space. In this case we need also to show that the value of the profile* $\rho_t(\cdot)$ *is fixed at the boundary. We leave this issue to Section A.4.*

- The case $\theta = 1$**:**

In this case we consider an arbitrary function $G : [0,1] \to \mathbb{R}$ which is two times continuously differentiable and we get

$$\begin{aligned} M_t^N(G) = \langle \pi_t^N, G\rangle - \langle \pi_0^N, G\rangle - \int_0^t \langle \pi_s^N, \frac{1}{2}\Delta_N G\rangle ds \\ - \frac{1}{2}\int_0^t \nabla_N^+ G(0)\eta_{sN^2}(1) - \nabla_N^- G(1)\eta_{sN^2}(N-1) ds \\ - \frac{\kappa}{2}\frac{N}{N-1}\int_0^t G\Big(\tfrac{1}{N}\Big)(\alpha - \eta_{sN^2}(1)) + G\Big(\tfrac{N-1}{N}\Big)(\beta - \eta_{sN^2}(N-1)) ds. \end{aligned}$$

In this regime Lemma A.4.2 is no longer valid. Nevertheless, by Remark A.3.4 we can replace $\eta_{sN^2}(1)$ (resp. $\eta_{sN^2}(N-1)$) by the average in a box around 1 (resp. $N-1$):

$$\overrightarrow{\eta}^{\varepsilon N}_{sN^2}(1) := \frac{1}{\varepsilon N}\sum_{x=1}^{1+\varepsilon N} \eta_{sN^2}(x), \quad \overleftarrow{\eta}^{\varepsilon N}_{sN^2}(N-1) := \frac{1}{\varepsilon N}\sum_{x=N-1}^{N-1-\varepsilon N} \eta_{sN^2}(x). \tag{2.8.4}$$

Here we note that the sum above goes from 1 to $1 + \lfloor \varepsilon N \rfloor$ but for sake of simplicity we write $1 + \varepsilon N$. By noting that $\overrightarrow{\eta}^{\varepsilon N}_{sN^2}(1) \sim \rho_s(0)$ (resp. $\overrightarrow{\eta}^{\varepsilon N}_{sN^2}(N-1) \sim \rho_s(1)$) - for details on this approximation see for example [1, 2] - and repeating the same arguments as above, we get to

$$\begin{aligned} \int_0^1 \rho_t(q)G(q) - \rho_0(q)G(q) dq - \int_0^t \int_0^1 \frac{1}{2}\Delta G(q)\rho_s(q) dq ds \\ - \frac{1}{2}\int_0^t \partial_q G(0)\rho_s(0) - \partial_q G(1)\rho_s(1) ds \\ + \frac{\kappa}{2}\int_0^t G(0)(\alpha - \rho_s(0)) - G(1)(\beta - \rho_s(1)) ds = 0. \end{aligned}$$

Again compare with the weak formulation given in (2.6.4) and note that the test function G does not depend on time.

- The case $\theta > 1$:

This regime is quite similar to the previous one. We consider again an arbitrary function $G : [0,1] \to \mathbb{R}$ which is two times continuously differentiable and we note that the last two terms at the right hand side of (2.8.3) vanish since $\theta > 1$. Then, repeating the same arguments as in the previous section and noting that Remark A.3.4 also applies to $\theta > 1$ we obtain at the end that

$$\int_0^1 \rho_t(q)G(q) - \rho_0(q)G(q)dq - \int_0^t \int_0^1 \frac{1}{2}\Delta G(q)\rho_s(q)dqds$$
$$-\frac{1}{2}\int_0^t \partial_q G(0)\rho_s(0) - \partial_q G(1)\rho_s(1)ds = 0.$$

Again compare with the weak formulation given in (2.6.4) and note that the test function G does not depend on time.

Remark 2.8.3. *Note that the parameter κ that appears in the boundary dynamics is only seen at the macroscopic level in the case $\theta = 1$ which corresponds to the heat equation with Robin boundary conditions.*

2.9 Tightness

In this section we show that the sequence of probability measures $\{\mathbb{Q}_N\}_{N\geq 1}$, defined in the beginning of Section 2.7, is tight in the Skorohod space $\mathscr{D}([0,T],\mathscr{M}_+)$. In order to do that, we invoke the Aldous's criterium which says that

Lemma 2.9.1. *A sequence $\{P_N\}_{N\geq 1}$ of probability measures defined on $\mathscr{D}([0,T],\mathscr{M}_+)$ is tight if these two conditions hold:*

a. *For every $t \in [0,T]$ and every $\varepsilon > 0$, there exists $K_\varepsilon^t \subset \mathscr{M}_+$ compact, such that*

$$\sup_{N\geq 1} P_N\Big(\pi_t \notin K_\varepsilon^t\Big) \leq \varepsilon,$$

b. *For every $\varepsilon > 0$*

$$\lim_{\gamma\to 0}\limsup_{N\to\infty}\sup_{\substack{\tau\in\mathscr{T}_T\\ \theta\leq\gamma}} P_N\Big(d(\pi_{\tau+\theta},\pi_\tau) > \varepsilon\Big) = 0,$$

where $\mathscr{T}_T$ denotes the set of stopping times with respect to the canonical filtration, bounded by T and d is the metric in the space $\mathscr{M}_+$.

By Proposition 1.7 of Chapter 4 in [16] it is enough to show that for every function G in a dense subset of $C([0,1])$, with respect to the uniform topology, the sequence of measures that corresponds to the real processes $\langle \pi_t^N, G\rangle$ is tight.

In our setting case, the first condition **a.** above translates by saying that:

$$\lim_{A\to+\infty}\lim_{N\to+\infty}\mathbb{P}_\mu\Big(|\langle \pi_t^N, G\rangle| > A\Big) = 0.$$

This is a consequence of Chebychev's inequality and the fact that for the exclusion type dynamics, the number of particles per site is at most one, we leave the details on this to the reader. So, it remains to show condition **b**. In this context and since we are considering the real process $\langle \pi_t^N, G\rangle$, the distance d above is the usual distance in $\mathbb{R}$. Then, we must show that for all $\varepsilon > 0$ and any function G in a dense subset of $C([0,1])$, with respect to the uniform topology, it holds that

$$\lim_{\delta\to 0}\limsup_{N\to\infty}\sup_{\tau\in\mathscr{T}_T,\bar{\tau}\leq\delta}\mathbb{P}_{\mu_N}\Big(\eta_\cdot : \Big|\langle \pi^N_{\tau+\bar{\tau}}, G\rangle - \langle \pi^N_\tau, G\rangle\Big| > \varepsilon\Big) = 0. \tag{2.9.1}$$

Above we assume that all the stopping times are bounded by T, thus, $\tau + \bar{\tau}$ should be understood as $(\tau + \bar{\tau}) \wedge T$.

Recall that it is enough to prove the assertion for functions G in a dense subset of $C([0,1])$ with respect to the uniform topology. We will use two different dense sets, namely the space $C^1([0,1])$ in the case $\theta < 1$ and the space $C^2([0,1])$ in the case $\theta \geq 1$, which are both dense in $C([0,1])$ with respect to the uniform topology. For that purpose, we split the proof according to $\theta \geq 1$ and $\theta < 1$. When $\theta \geq 1$ we prove (2.9.1) directly for functions $G \in C^2([0,1])$ and we conclude that the sequence is tight. For $\theta < 1$, we prove (2.9.1) first for functions $G \in C_c^2(0,1)$ and then we extend it, by a L^1 approximation procedure which is explained below, to functions $G \in C^1([0,1])$.

Recall from (2.8.3) that $M_t^N(G)$ is a martingale with respect to the natural filtration $\{\mathscr{F}_t\}_{t\geq 0}$. Then

$$\begin{aligned}
&\mathbb{P}_{\mu_N}\Big(\eta_\cdot : \Big|\langle \pi^N_{\tau+\bar{\tau}}, G\rangle - \langle \pi^N_\tau, G\rangle\Big| > \varepsilon\Big)\\
=&\mathbb{P}_{\mu_N}\Big(\eta_\cdot : \Big|M^N_\tau(G) - M^N_{\tau+\bar{\tau}}(G) + \int_\tau^{\tau+\bar{\tau}} N^2\mathscr{L}_N\langle \pi_s^N, G\rangle ds\Big| > \varepsilon\Big)\\
\leq&\mathbb{P}_{\mu_N}\Big(\eta_\cdot : \Big|M^N_\tau(G) - M^N_{\tau+\bar{\tau}}(G)\Big| > \frac{\varepsilon}{2}\Big)\\
+&\mathbb{P}_{\mu_N}\Big(\eta_\cdot : \Big|\int_\tau^{\tau+\bar{\tau}} N^2\mathscr{L}_N\langle \pi_s^N, G\rangle ds\Big| > \frac{\varepsilon}{2}\Big).
\end{aligned}$$

Applying Chebychev's inequality (resp. Markov's inequality) in the first (resp. second) term on the right hand side of last inequality, we can bound the previous expression from above by

$$\frac{2}{\varepsilon^2}\mathbb{E}_{\mu_N}\Big[\Big(M_\tau^N(G)-M_{\tau+\bar\tau}^N(G)\Big)^2\Big]+\frac{2}{\varepsilon}\mathbb{E}_{\mu_N}\Big[\Big|\int_\tau^{\tau+\bar\tau}N^2\mathscr{L}_N\langle\pi_s^N,G\rangle ds\Big|\Big].$$

Therefore, in order to prove (2.9.1) it is enough to show that

$$\lim_{\delta\to 0}\limsup_{N\to\infty}\sup_{\tau\in\mathscr{T}_T,\bar\tau\le\delta}\mathbb{E}_{\mu_N}\Big[\Big|\int_\tau^{\tau+\bar\tau}N^2\mathscr{L}_N\langle\pi_s^N,G\rangle ds\Big|\Big]=0 \tag{2.9.2}$$

and

$$\lim_{\delta\to 0}\limsup_{N\to\infty}\sup_{\tau\in\mathscr{T}_T,\bar\tau\le\delta}\mathbb{E}_{\mu_N}\Big[\big(M_\tau^N(G)-M_{\tau+\bar\tau}^N(G)\big)^2\Big]=0. \tag{2.9.3}$$

Let us start by proving (2.9.2). Given a test function G, we will show that there exists a contant C such that

$$N^2\mathscr{L}_N(\langle\pi_s^N,G\rangle)\le C \tag{2.9.4}$$

for any $s\le T$. We start with the case $\theta\ge 1$. For that purpose, recall (2.8.2). Note that, since $|\eta_{sN^2}(x)|\le 1$ for all $s\in[0,t]$ and since $G\in C^2([0,1])$, we have that

$$\Big|\langle\pi_s^N,\Delta_N G\rangle+\nabla_N^+G(0)\eta_{sN^2}(1)-\nabla_N^-G(1)\eta_{sN^2}(N-1)\Big|\le 2\|G''\|_\infty+2\|G\|_\infty$$

and

$$\begin{aligned}\Big|\kappa N^{1-\theta}G\Big(\tfrac{1}{N}\Big)(\alpha-\eta_{sN^2}(1))+\kappa N^{1-\theta}G\Big(\tfrac{N-1}{N}\Big)(\beta-\eta_{sN^2}(N-1))\Big|&\le 4\kappa N^{1-\theta}\|G\|_\infty\\&\le 4\kappa\|G\|_\infty.\end{aligned}$$

This proves (2.9.4) for the case $\theta\ge 1$. In the case $\theta<1$, we take $G\in C_c^2([0,1])$ and we see that in this case (2.8.2) reduces to $\langle\pi_s^N,\frac{1}{2}\Delta_N G\rangle$ whose absolute value is bounded from above by $\|G''\|_\infty$ and this proves (2.9.4) for the case $\theta<1$.

Let us now prove (2.9.3). Applying Dynkin's formula with F given by (2.8.1) we get that

$$\big(M_t^N(G)\big)^2-\int_0^t N^2\big[\mathscr{L}_N\langle\pi_s^N,G\rangle^2-2\langle\pi_s^N,G\rangle\mathscr{L}_N\langle\pi_s^N,G\rangle\big]ds, \tag{2.9.5}$$

is a martingale with respect to the natural filtration $\{\mathscr{F}_t\}_{t\geq 0}$. A simple computation shows that

$$\begin{aligned}&N^2\big[\mathscr{L}_{N,0}\langle \pi_s^N, G\rangle^2 - 2\langle \pi_s^N, G\rangle \mathscr{L}_{N,0}\langle \pi_s^N, G\rangle\big]\\ =&\frac{1}{2N^2}\sum_{x=1}^{N-2}(\eta_{sN^2}(x)-\eta_{sN^2}(x+1))^2(\nabla_N G(\tfrac{x}{N}))^2\end{aligned}$$

and by using the fact that $|\eta_{sN^2}(x)| \leq 1$ for all $s \in [0,t]$ last expression is bounded from above by $\frac{2}{N}\|G'\|_\infty$. On the other hand, we also have that

$$\begin{aligned}&N^2\big[\mathscr{L}_{N,b}\langle \pi_s^N, G\rangle^2 - 2\langle \pi_s^N, G\rangle \mathscr{L}_{N,b}\langle \pi_s^N, G\rangle\big]\\ =&\frac{\kappa}{2N^\theta}\Big[c_1(\eta_{sN^2},\alpha)G(\tfrac{1}{N})^2 + c_{N-1}(\eta_{sN^2},\beta)G(\tfrac{N-1}{N})^2\Big]\end{aligned}$$

and by using the fact that $|\eta_{sN^2}(x)| \leq 1$ for all $s \in [0,t]$ last expression is bounded from above by $\frac{4\kappa}{N^\theta}\|G\|_\infty^2$.

This ends the proof of tightness in the case $\theta \geq 1$, since $C^2([0,1])$ is a dense subset of $C([0,1])$ with respect to the uniform topology. Nevertheless, for $\theta < 1$, since we considered functions $G \in C_c^2(0,1)$, last display is equal to zero. Therefore, we have proved (2.9.2) and (2.9.3), and thus (2.9.1), but for functions $G \in C_c^2(0,1)$ and, as mentioned above, we need to extend this result to functions in $C^1([0,1])$. To accomplish that, we take a function $G \in C^1([0,1]) \subset L^1([0,1])$, and we take a sequence of functions $\{G_k\}_{k\geq 0} \in C_c^2(0,1)$ converging to G, with respect to the L^1-norm, as $k \to \infty$. Now, since the probability in (2.9.1) is less or equal than

$$\begin{aligned}&\mathbb{P}_{\mu_N}\Big(\eta_\cdot : \Big|\langle \pi^N_{\tau+\bar\tau}, G_k\rangle - \langle \pi^N_\tau, G_k\rangle\Big| > \frac{\varepsilon}{2}\Big)\\ +\quad &\mathbb{P}_{\mu_N}\Big(\eta_\cdot : \Big|\langle \pi^N_{\tau+\bar\tau}, G-G_k\rangle - \langle \pi^N_\tau, G-G_k\rangle\Big| > \frac{\varepsilon}{2}\Big)\end{aligned}$$

and since G_k has compact support, from the computation above, it remains only to check that the last probability vanishes as $N \to \infty$ and then $k \to \infty$. For that purpose, we use the fact that

$$\Big|\langle \pi^N_{\tau+\bar\tau}, G-G_k\rangle - \langle \pi^N_\tau, G-G_k\rangle\Big| \leq \frac{2}{N}\sum_{x\in\Lambda_N}\Big|(G-G_k)(\tfrac{x}{N})\Big|, \tag{2.9.6}$$

and we use the estimate

$$\begin{aligned}\frac{1}{N}\sum_{x\in\Lambda_N}\left|(G-G_k)(\tfrac{x}{N})\right| &\le \sum_{x\in\Lambda_N}\int_{\frac{x}{N}}^{\frac{x+1}{N}}\left|(G-G_k)(\tfrac{x}{N})-(G-G_k)(q)\right|dq \\ &+\int_0^1|(G-G_k)(q)|dq \\ &\le \frac{1}{N}\|(G-G_k)'\|_\infty+\int_0^1|(G-G_k)(q)|dq.\end{aligned}$$

The result follows by first taking $N\to\infty$ and then $k\to\infty$.

2.10 The limit point

Here, we prove at first that all limit points $\mathbb{Q}$ of the sequence $\{\mathbb{Q}_N\}_{N\ge1}$ are concentrated on measures absolutely continuous with respect to the Lebesgue measure, that are equal to $g(q)dq$ at the initial time and finally that $\mathbb{Q}$ is concentrated on trajectories of measures satisfying $\pi_t(dq)=\rho_t(q)dq$, where $\rho_t(\cdot)$ is the weak solution of the corresponding hydrodynamic equation. Let $\mathbb{Q}$ be a limit point of $\{\mathbb{Q}_N\}_{N\ge1}$.

2.10.1 Characterization of absolutely continuity

We start by showing that $\mathbb{Q}$ is concentrated on measures which are absolutely continuous with respect to the Lebesgue measure. Fix a continuous function $G:[0,1]\to\mathbb{R}$. Since

$$\sup_{t\in[0,T]}|\langle\pi_t^N,G\rangle|\le\frac{1}{N}\sum_{x\in\Lambda_N}|G(\tfrac{x}{N})|,$$

which is a consequence of the fact of having at most one particle per site, the function that associates to each trajectory $\pi_\cdot$, $\sup_{t\in[0,T]}|\langle\pi_t,G\rangle|$ is continuous. As a consequence, all limit points are concentrated in trajectories π_t such that

$$|\langle\pi_t,G\rangle|\le\int_0^1|G(q)|dq.$$

In order to show that the measure π_t is absolutely continuous with respect to the Lebesgue measure, that we denote by Leb, we have to show that for each set A such that $Leb(A)=0$, then $\pi_t(A)=0$. With this purpose, we use last estimate for a sequence of continuous functions $\{G_N\}_{N\geq 1}$ that converge to the indicator function over the set A and the result follows. Concluding, we have just proved that

$$\mathbb{Q}\Big(\pi_\cdot : \pi_t(dq) = \pi(t,q)dq, \forall t \in [0,T]\Big) = 1$$

i.e. $\pi_t(dq)$ is absolutely continuous with respect to the Lebesgue measure with density $\pi(t,q)$.

2.10.2 Characterization of the initial measure

Here we show that $\mathbb{Q}$ is concentrated on a Dirac measure equal to $g(q)dq$ at time 0. For that purpose, fix $\varepsilon > 0$. From the results of Section 2.9, we know, from the weak convergence over a subsequence and Portmanteau's Theorem, that:

$$\begin{aligned}
&\mathbb{Q}\Big(\Big|\frac{1}{N}\sum_{x\in\Lambda_N} G(\tfrac{x}{N})\eta_0(x) - \int_0^1 G(q)g(q)dq\Big| > \varepsilon\Big)\\
&\leq \liminf_{K\to+\infty} \mathbb{Q}_{N_k}\Big(\Big|\frac{1}{N}\sum_{x\in\Lambda_N} G(\tfrac{x}{N})\eta_0(x) - \int_0^1 G(q)g(q)dq\Big| > \varepsilon\Big)\\
&= \liminf_{K\to+\infty} \mu_{N_k}\Big(\Big|\frac{1}{N}\sum_{x\in\Lambda_N} G(\tfrac{x}{N})\eta(x) - \int_0^1 G(q)g(q)dq\Big| > \varepsilon\Big).
\end{aligned}$$

This last limit is equal to zero, by the hypothesis of μ_N being associated to the profile $g(\cdot)$, see Definition 2.7.1. This shows that

$$\mathbb{Q}\Big(\pi_\cdot : \pi_0(dq) = g(q)dq\Big) = 1.$$

2.10.3 Characterization of the density $\pi(t,q)$

From Section 2.10.1 we know that all limit points $\mathbb{Q}$ of the sequence sequence $\{\mathbb{Q}_N\}_{N\in\mathbb{N}}$ are concentrated on trajectories $\pi_t(dq)$ which are absolutely continuous with respect to the Lebesgue measure, that is, $\pi_t(dq) = \pi(t,q)dq$. Moreover, from the previous section we also know that all limit points $\mathbb{Q}$ of the sequence $\{\mathbb{Q}_N\}_{N\in\mathbb{N}}$ are such that the initial trajectory is a Dirac measure equal to $g(q)dq$. Now we prove that all limit points are concentrated on trajectories of measures of the form $\rho_t(q)dq$, that is we are going to show that $\pi(t,q) = \rho_t(q)$ and that $\rho_t(\cdot)$ is a weak solution of the corresponding hydrodynamic equation.

For that purpose, let $\mathbb{Q}$ be a limit point of the sequence $\{\mathbb{Q}_N\}_{N\geq 1}$, whose existence follows from the computations of Section 2.9 and assume, without lost of generality, that $\{\mathbb{Q}_N\}_{N\geq 1}$ converges to $\mathbb{Q}$, as $N \to +\infty$.

Proposition 2.10.1. *If $\mathbb{Q}$ is a limit point of $\{\mathbb{Q}_N\}_{N\in\mathbb{N}}$ then*

$$\mathbb{Q}\Big(\pi_\cdot : F_\theta = 0, \forall\, t \in [0,T],\ \forall G \in C_\theta\Big) = 1,$$

where

$$F_\theta = \begin{cases} F^c_{Dir},\ \text{if } \theta < 0, \\ F_{Dir},\ \text{if } \theta \in [0,1), \\ F_{Rob},\ \text{if } \theta \geq 1, \end{cases} \qquad \text{and} \qquad C_\theta = \begin{cases} C^{1,2}_c([0,T]\times[0,1]),\ \text{if } \theta < 0, \\ C^{1,2}_0([0,T]\times[0,1]),\ \text{if } \theta \in [0,1), \\ C^{1,2}([0,T]\times[0,1]),\ \text{if } \theta \geq 1. \end{cases}$$

Proof. We consider the case $\theta \geq 1$. Note that we need to verify, for $\delta > 0$ and $G \in C^{1,2}([0,T]\times[0,1])$, that

$$\mathbb{Q}\left(\pi_\cdot \in \mathscr{D}([0,T],\mathscr{M}^+) : \sup_{0\leq t\leq T} |F_{Rob}| > \delta\right) = 0, \tag{2.10.1}$$

Recall F_{Rob} from (2.6.6) and note that, due to the terms that involve $\rho_s(1)$ and $\rho_s(0)$ and that appear in F_{Rob}, the set inside the probability in (2.10.1) is not an open set in the Skorohod space, and as a consequence we cannot use directly Portmanteau's Theorem. To avoid this difficulty, we fix $\varepsilon > 0$ and we consider two approximations of the identity given by

$$\iota^0_\varepsilon(q) = \frac{1}{\varepsilon}1_{(0,\varepsilon)}(q) \quad \text{and} \quad \iota^1_\varepsilon(q) = \frac{1}{\varepsilon}1_{(1-\varepsilon,1)}(q) \tag{2.10.2}$$

and we sum and subtract to $\rho_s(0)$ and to $\rho_s(1)$ the mean

$$\langle \pi_s, \iota^0_\varepsilon\rangle = \tfrac{1}{\varepsilon}\int_0^\varepsilon \rho_s(q)dq \quad \text{and} \quad \langle \pi_s, \iota^1_\varepsilon\rangle = \tfrac{1}{\varepsilon}\int_{1-\varepsilon}^\varepsilon \rho_s(q)dq, \tag{2.10.3}$$

respectively. Above we used the fact that $\mathbb{Q}$ is concentrated on trajectories $\pi_t(dq)$ which are absolutely continuous with respect to the Lebesgue measure: $\pi_t(dq) = \rho_t(q)dq$. Thus, we bound the probability in (2.10.1) from above by the sum of the following terms

$$\begin{aligned}&\mathbb{Q}\Bigg(\sup_{0\leq t\leq T}\Bigg|\int_0^1 \rho_t(q)G_t(q)\,dq - \int_0^1 \rho_0(q)G_0(q)\,dq \\ &- \int_0^t\int_0^1 \rho_s(q)\Big(\frac{1}{2}\Delta + \partial_s\Big)G_s(q)\,dq ds - \frac{1}{2}\int_0^t G_s(0)\alpha + G_s(1)\beta\,ds \\ &+ \frac{1}{2}\int_0^t \langle \pi_s, \iota^1_\varepsilon\rangle\Big(\partial_q G_s(1) + G_s(1)\Big)\,ds - \frac{1}{2}\int_0^t \langle \pi_s, \iota^0_\varepsilon\rangle\Big(\partial_q G_s(0) - G_s(0)\Big)\,ds\Bigg| > \frac{\delta}{4}\Bigg),\end{aligned} \tag{2.10.4}$$

$$\mathbb{Q}\left(\left|\int_0^1 (\rho_0(q)-g(q))G_0(q)\,dq\right|>\frac{\delta}{4}\right), \tag{2.10.5}$$

$$\sum_{k\in\{0,1\}}\mathbb{Q}\left(\sup_{0\le t\le T}\left|\frac{1}{2}\int_0^t(\rho_s(k)-\langle\pi_s,\iota_\varepsilon^k\rangle)\big[G_s(k)-\partial_q G_s(k)\big]\,ds\right|>\frac{\delta}{4}\right). \tag{2.10.6}$$

and we note that the terms in (2.10.6) converge to 0 as $\varepsilon\to 0$ since we are comparing $\rho_s(0)$ and $\rho_s(1)$ with the averages (2.10.3) around 0 and 1, respectively. Moreover, (2.10.5) is equal to zero since $\mathbb{Q}$ is a limit point of $\{\mathbb{Q}_N\}_{N\in\mathbb{N}}$ and $\mathbb{Q}_N$ is induced by a measure μ_N which is associated to the profile $g(\cdot)$. Note that in (2.10.4) we still cannot use Portmanteau's Theorem, since the functions ι_ε^0 and ι_ε^1 are not continuous. Nevertheless, by approximating each one of these functions by continuous functions in such a way that the error vanishes as $\varepsilon\to 0$ then, from Proposition A.3 of [11] we can use Portmanteau's Theorem and bound (2.10.4) from above by

$$\begin{aligned}&\liminf_{N\to\infty}\mathbb{Q}_N\Big(\sup_{0\le t\le T}\Big|\int_0^1\rho_t(q)G_t(q)\,dq-\int_0^1\rho_0(q)G_0(q)\,dq\\&-\int_0^t\int_0^1\rho_s(q)\Big(\frac{1}{2}\Delta+\partial_s\Big)G_s(q)\,dqdsv-\frac{1}{2}\int_0^t G_s(0)\alpha+G_s(1)\beta\,ds\\&-\frac{1}{2}\int_0^t\langle\pi_s,\iota_\varepsilon^0\rangle\Big(\partial_q G_s(0)-G_s(0)\Big)ds+\frac{1}{2}\int_0^t\langle\pi_s,\iota_\varepsilon^1\rangle\Big(\partial_q G_s(1)+G_s(1)\Big)ds\Big|>\frac{\delta}{2^4}\Big).\end{aligned} \tag{2.10.7}$$

Summing and subtracting $\displaystyle\int_0^t N^2\mathscr{L}_N\langle\pi_s^N,G_s\rangle ds$ to the term inside the supremum in (2.10.7), recalling (2.8.3) and (2.8.4), the definition of $\mathbb{Q}_N$, we bound (2.10.7) from above by the sum of the next two terms

$$\liminf_{N\to\infty}\mathbb{P}_{\mu_N}\left(\sup_{0\le t\le T}\big|M_t^N(G)\big|>\frac{\delta}{2^5}\right), \tag{2.10.8}$$

and

$$\begin{aligned}&\liminf_{N\to\infty}\mathbb{P}_{\mu_N}\left(\sup_{0\le t\le T}\Big|\int_0^t N^2\mathscr{L}_N\langle\pi_s^N,G_s\rangle\,ds-\int_0^t\int_0^1\rho_s(q)\frac{1}{2}\Delta G_s(q)\,dqds\right.\\&-\frac{1}{2}\int_0^t\overrightarrow{\eta}^{\varepsilon N}_{sN^2}(1)\Big(\partial_q G_s(0)-G_s(0)\Big)ds+\frac{1}{2}\int_0^t\overleftarrow{\eta}^{\varepsilon N}_{sN^2}(N-1)\Big(\partial_q G_s(1)+G_s(1)\Big)ds\\&\left.-\frac{1}{2}\int_0^t G_s(0)\alpha+G_s(1)\beta\,ds\Big|>\frac{\delta}{2^5}\right).\end{aligned} \tag{2.10.9}$$

Doob's inequality together with the computations right below (2.9.5) show that (2.10.8) goes to 0 as $N \to \infty$. Finally, (2.10.9) can be rewritten as

$$\liminf_{N\to\infty} \mathbb{P}_{\mu_N}\Big(\sup_{0\le t\le T}\Big|\int_0^t N^2\mathscr{L}_N\langle \pi_s^N, G_s\rangle\, ds - \int_0^t \langle \pi_s^N, \frac{1}{2}\Delta G_s\rangle\, ds$$
$$-\frac{1}{2}\int_0^t \overrightarrow{\eta}^{\varepsilon N}_{sN^2}(1)\Big(\partial_q G_s(0) - G_s(0)\Big)\, ds + \frac{1}{2}\int_0^t \overleftarrow{\eta}^{\varepsilon N}_{sN^2}(N-1)\Big(\partial_q G_s(1) + G_s(1)\Big)\, ds$$
$$-\frac{1}{2}\int_0^t G_s(0)\alpha + G_s(1)\beta\, ds\Big| > \frac{\delta}{2^5}\Big). \tag{2.10.10}$$

Now, from (2.8.2) we can bound from above the probability in (2.10.10) by the sum of the following terms

$$\mathbb{P}_{\mu_N}\left(\sup_{0\le t\le T}\Big|\frac{1}{N}\int_0^t \sum_{x\in\Lambda_N}\frac{1}{2}\Delta_N G_s(\tfrac{x}{N})\eta_{sN^2}(x)ds - \int_0^t \Big\langle \pi_s^N, \frac{1}{2}\Delta G_s\Big\rangle\, ds\Big| > \frac{\delta}{2^6}\right), \tag{2.10.11}$$

$$\mathbb{P}_{\mu_N}\left(\sup_{0\le t\le T}\Big|\frac{1}{2}\int_0^t \nabla_N^+ G_s(0)\eta_{sN^2}(1) - \overrightarrow{\eta}^{\varepsilon N}_{sN^2}(1)\partial_q G_s(0)\, ds\Big| > \frac{\delta}{2^6}\right), \tag{2.10.12}$$

and

$$\mathbb{P}_{\mu_N}\left(\sup_{0\le t\le T}\Big|\frac{1}{2}\int_0^t \kappa N^{1-\theta}G_s\Big(\tfrac{1}{N}\Big)(\alpha - \eta_{sN^2}(1)) - G_s(0)(\alpha - \overrightarrow{\eta}^{\varepsilon N}_{sN^2}(1))ds\Big| > \frac{\delta}{2^6}\right) \tag{2.10.13}$$

and two other terms which are very similar to the two previous ones but related to the action of the right boundary dynamics given by $\mathscr{L}_{N,b}^{N-1}$. Applying a Taylor expansion on the test function G it is easy to show that (2.10.11) goes to 0 as $N \to \infty$. Also by Taylor expansion, (2.10.12) can be bounded from above by

$$\mathbb{P}_{\mu_N}\left(\sup_{0\le t\le T}\Big|\int_0^t \partial_q G_s(0)(\eta_{sN^2}(1) - \overrightarrow{\eta}^{\varepsilon N}_{sN^2}(1))ds\Big| > \frac{\delta}{2^8}\right). \tag{2.10.14}$$

plus a term that vanishes as $N \to \infty$. Using Lemma A.3.2 we see that (2.10.14) vanishes as $N \to \infty$. The term (2.10.13) can be estimated using exactly the same argument that we just used, that is: Taylor expansion on G plus Lemma A.3.2. For the terms related to the right boundary the argument is the same and with this we finish the proof.

We leave the other cases, namely $\theta < 1$ for the reader. These cases are even simpler than the previous one and for the interested reader we refer to, for example, [1, 2].

□

2.11 Hydrostatic limit

In this section we prove that the hydrodynamic limit holds when we start the system from the stationary measure μ_{ss}, see Section 2.4. By looking at the statement of Theorem 2.7.2 we see that in fact to conclude we only need to show the next result.

Proposition 2.11.1. *Let μ_{ss} be the stationary measure for the Markov process $\{\eta_{tN^2} : t \geq 0\}$ with generator $N^2\mathscr{L}_N$. Then, μ_{ss} is associated to the profile $\bar{\rho} : [0,1] \to [0,1]$ given on $q \in (0,1)$ by* (2.5.5)*, which is a stationary solution of the corresponding hydrodynamic equation, see* (2.6.10) *and* (2.6.11).

Proof. Recall from (2.7.2), that we need to prove:

$$\lim_{N\to\infty} \mu_{ss}\Big(\eta \in \Omega_N : \Big|\frac{1}{N}\sum_{x\in\Lambda_N} G\left(\tfrac{x}{N}\right)\eta(x) - \int_0^1 G(q)\rho_0(q)dq\Big| > \delta\Big) = 0.$$

By Markov's and triangular inequalities, we bound the previous probability from above by

$$\begin{aligned}
&\frac{1}{\delta}E_{\mu_{ss}}\Big[\Big|\frac{1}{N}\sum_{x\in\Lambda_N} G\left(\tfrac{x}{N}\right)\Big(\eta(x) - \rho^N_{ss}(x)\Big)\Big| \\
&+\Big|\frac{1}{N}\sum_{x\in\Lambda_N} G\left(\tfrac{x}{N}\right)\rho^N_{ss}(x) - \int_0^1 G(q)\bar{\rho}(q)dq\Big|\Big] \\
&\leq \frac{1}{\delta}E_{\mu_{ss}}\Big[\Big|\frac{1}{N}\sum_{x\in\Lambda_N} G\left(\tfrac{x}{N}\right)\Big(\eta(x) - \rho^N_{ss}(x)\Big)\Big|\Big] \\
&+\frac{1}{\delta}\Big|\frac{1}{N}\sum_{x\in\Lambda_N} G\left(\tfrac{x}{N}\right)\rho^N_{ss}(x) - \int_0^1 G(q)\bar{\rho}(q)dq\Big|.
\end{aligned} \tag{2.11.1}$$

The last term can be bounded from above by

$$\frac{1}{\delta}\Big|\frac{1}{N}\sum_{x\in\Lambda_N} G\left(\tfrac{x}{N}\right)\Big(\rho^N_{ss}(x) - \bar{\rho}\left(\tfrac{x}{N}\right)\Big)\Big| + \frac{1}{\delta}\Big|\frac{1}{N}\sum_{x\in\Lambda_N} G\left(\tfrac{x}{N}\right)\bar{\rho}\left(\tfrac{x}{N}\right) - \int_0^1 G(q)\bar{\rho}(q)dq\Big|.$$

The term at the left hand side of last expression is bounded from above by

$$\frac{1}{\delta}\frac{1}{N}\sum_{x\in\Lambda_N}\Big|G\left(\tfrac{x}{N}\right)\Big|\Big|\rho^N_{ss}(x) - \bar{\rho}\left(\tfrac{x}{N}\right)\Big| \leq \frac{\|G\|_\infty}{\delta}\max_{x\in\Lambda_N}\Big|\rho^N_{ss}(x) - \bar{\rho}\left(\tfrac{x}{N}\right)\Big|$$

where from (2.5.4) it vanishes as $N \to \infty$, while the term at the right hand side also vanishes as $N \to \infty$ since we compare the Riemann sum with the corresponding converging integral. To finish the proof it remains to analyse the third term in (2.11.1). By the Cauchy-Schwarz's inequality the expectation appearing in that term can bounded from above by

$$
\begin{aligned}
&\Big(\Big|\frac{1}{N^2}\sum_{x\in\Lambda_N} G\left(\tfrac{x}{N}\right)\mathbb{E}_{\mu_{ss}}\Big[(\eta(x)-\rho^N_{ss}(x))^2\Big]\\
&+\frac{2}{N}\sum_{x<y} G\left(\tfrac{x}{N}\right)G\left(\tfrac{y}{N}\right)E_{\mu_{ss}}\Big[(\eta(x)-\rho^N_{ss}(x))(\eta(y)-\rho^N_{ss}(y))\Big]\Big)^{\frac{1}{2}}\\
&\leq\Big(\frac{C\|G\|_\infty}{N}+2\|G\|_\infty\max_{x<y}\varphi^N_{ss}(x,y)\Big)^{\frac{1}{2}}.
\end{aligned}
$$

From (2.4.12) the previous expression vanishes as $N\to\infty$. This finishes the proof. □

Chapter 3

Symmetric exclusion with long jumps in contact with reservoirs

3.1 The model

In this chapter we want to generalize the results of the previous chapter to the case where particles can give jumps arbitrarily large. As in the previous chapter, the bulk consists in the set of points $\Lambda_N = \{1, \cdots, N-1\}$ and we artificially add two end points $x = 0$ and $x = N$. Now, we explain the dynamics of the models we consider and we start by describing the conditions on the jump rate. For that purpose, let $p : \mathbb{Z} \times \mathbb{Z} \to [0,1]$ be a transition probability such that $p(x,y) = p(y-x)$ and which is symmetric. We are going to discuss two cases: the first one, when $p(\cdot)$ has finite variance and the second one when $p(\cdot)$ has infinite variance. Note that since $p(\cdot)$ is symmetric it has mean zero, that is: $\sum_{z\in\mathbb{Z}} zp(z) = 0$. We denote $m = \sum_{z\geq 1} zp(z)$. As an example we consider $p(\cdot)$ given by $p(0) = 0$ and

$$p(z) = \frac{c_\gamma}{|z|^{\gamma+1}}, \tag{3.1.1}$$

for $z \neq 0$, where c_γ is a normalizing constant. For simplicity of the presentation we stick to this choice of $p(\cdot)$ along this chapter but we note that many of our results are true, in the case where $p(\cdot)$ has finite variance, in a more general setting where we only assume $p(\cdot)$ to be translation invariant and mean zero.

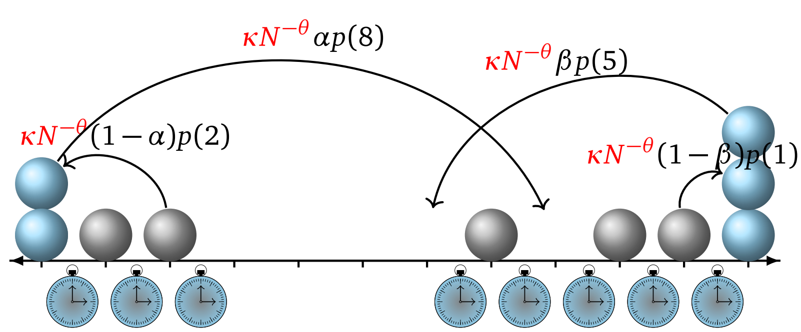
We consider the process in contact with stochastic reservoirs at the left and the right of the bulk. We fix four parameters $\alpha, \beta \in [0,1]$, $\kappa > 0$ and $\theta \in \mathbb{R}$, so that particles can get in the bulk of the system from the site $x = 0$ to any site $y \in \Lambda_N$ at rate $\alpha\kappa N^{-\theta} p(y)$ or leave the bulk from any site $y \in \Lambda_N$ to the site $x = 0$ at rate $(1-\alpha)\kappa N^{-\theta} p(y)$; and particles can get in the bulk to any site

$y \in \Lambda_N$ from the site $x = N$ at rate $\beta\kappa N^{-\theta} p(N-y)$ or leave the bulk from any site $y \in \Lambda_N$ to the site $x = N$ at rate $(1-\beta)\kappa N^{-\theta} p(N-y)$.

We define the dynamics of the process in the following way. We start with the bulk dynamics. Each pair of sites of the bulk $\{x, y\} \subset \Lambda_N$ carries a Poisson process of intensity $p(y-x)/2$. Poisson processes associated to different bonds are independent. If for the configuration η, the clock associated to the bound $\{x, y\}$ rings, then we exchange the value of the occupation variables $\eta(x)$ and $\eta(y)$ at rate $p(y-x)/2$. Now we explain the dynamics at the boundary. Each pair of sites $\{0, x\}$ with $x \in \Lambda_N$ carries two Poisson processes, all of them being independent. If for the configuration η, the clock associated to the Poisson process of the bond $\{0, x\}$ rings, then we change the value $\eta(x)$ into $1-\eta(x)$ with rate $\kappa N^{-\theta} p(x)[(1-\alpha)\eta(x) + \alpha(1-\eta(x))]$. At the right boundary the dynamics is similar but instead of α the intensity is given by β. Observe that the reservoirs ($x = 0$ and $x = N$) add and remove particles on all the sites of the bulk Λ_N, and not only at the boundaries $x = 1$ and $x = N-1$ as happened in the model of Chapter 2, but with a rate that decreases as the distance from the corresponding reservoir increases. We remark that as in the previous chapter, we could do another interpretation of the previous dynamics at the boundary, as follows. Particles can either be created or annihilated at any site $x \in \Lambda_N$ according to the following rates:

- from the left reservoir, from $x = 0$ to $y \in \Lambda_N$:
  - creation rate: $\alpha\kappa N^{-\theta} p(y)$,
  - annihilation rate: $(1-\alpha)\kappa N^{-\theta} p(y)$.
- from the right reservoir, from $x = N-1$ to $y \in \Lambda_N$:
  - creation rate: $\beta\kappa N^{-\theta} p(N-y)$,
  - annihilation rate: $(1-\beta)\kappa N^{-\theta} p(N-y)$.

Let us see an illustration of the dynamics just described with $N = 11$ and the configuration $\eta = (1, 1, 0, 0, 0, 0, 1, 0, 1, 1)$:



The infinitesimal generator of the process is given by

$$\mathscr{L}_N = \mathscr{L}_{N,0} + \mathscr{L}_{N,b}, \tag{3.1.2}$$

where $\mathscr{L}_{N,0}$ and $\mathscr{L}_{N,b}$ act on functions $f : \Omega_N \to \mathbb{R}$ as

$$\begin{aligned}
(\mathscr{L}_{N,0}f)(\eta) &= \frac{1}{2}\sum_{x,y\in\Lambda_N} p(x-y)[f(\eta^{x,y})-f(\eta)],\\
(\mathscr{L}_{N,b}f)(\eta) &= \frac{\kappa}{N^\theta}\sum_{y\in\{0,N\}}\sum_{x\in\Lambda_N} p(y-x)c_x(\eta, r(y))[f(\eta^x)-f(\eta)]
\end{aligned} \tag{3.1.3}$$

where the configurations $\eta^{x,y}$ and η^x have been defined in (2.3.3), the rates $c_x(\eta, r(y))$ have been defined in (2.3.4) and $r(0) = \alpha$ and $r(N) = \beta$.

We consider the Markov process speeded up in the time scale $t\Theta(N)$ and note that $\{\eta_{t\Theta(N)} : t \geq 0\}$ has infinitesimal generator given by $\Theta(N)\mathscr{L}_N$. Although $\eta_{t\theta(N)}$ depends on α, β and θ, we shall omit these index in order to simplify notation.

As in Section 2.4 we can prove that the Bernoulli product measures ν_ρ^N as defined in (2.4.1) are reversible when we consider $\alpha = \beta = \rho$. The proof is quite similar to the one given in Lemma 2.4.1 and for that reason it is omitted.

In the next section we analyse the case where $p(\cdot)$ has finite variance and we denote it by σ^2, so that

$$\sigma^2 := \sum_{z\in\mathbb{Z}} z^2 p(z) < \infty.$$

As an example we consider $p(\cdot)$ as in (3.1.1), that is $p(0) = 0$ and

$$p(z) = \frac{c_\gamma}{|z|^{\gamma+1}},$$

for $z \neq 0$, where c_γ is a normalizing constant and we take $\gamma > 2$, so that $p(\cdot)$ has finite variance. For simplicity of the presentation we stick to this choice of $p(\cdot)$ whenever we mention to the case where $p(\cdot)$ has finite variance but we note that many of our results are true in the more general setting where we just assume $p(\cdot)$ to be translation invariant, mean zero and with finite variance.

Remark 3.1.1. *We note that for the choice of p with $p(1) = \frac{1}{2} = p(-1)$ the dynamics described above coincides with the one of the first chapter. In that sense many of the results that we will derive here are a generalization of those obtained before.*

In Section 3.3 we analyse the case where $p(\cdot)$ is as in (3.1.1) but we consider $\gamma \in (1, 2)$ so that $p(\cdot)$ is mean zero but with infinite variance.

3.2 The finite variance case

3.2.1 Hydrodynamic equations: finite variance

Recall the notation introduced in Section 2.6. We can now give the definition of the weak solutions of the hydrodynamic equations that will be derived in this chapter when $p(\cdot)$ is assumed to have finite variance. In what follows $g : [0,1] \to [0,1]$ is a measurable function and it is the initial condition of all the partial differential equations that we define below, that is $\rho_0(q) = g(q)$, for all $q \in (0,1)$.

Definition 3.2.1. *Let $\hat{\sigma} \geq 0$ and $\hat{\kappa} \geq 0$ be some parameters. We say that $\rho : [0,T] \times [0,1] \to [0,1]$ is a weak solution of the reaction-diffusion equation with Dirichlet boundary conditions*

$$\begin{cases} \partial_t \rho_t(q) = \frac{\hat{\sigma}^2}{2} \Delta \rho_t(q) + \hat{\kappa} \Big\{ \frac{\alpha - \rho_t(q)}{q^{\gamma+1}} + \frac{\beta - \rho_t(q)}{(1-q)^{\gamma+1}} \Big\}, & (t,q) \in (0,T] \times (0,1), \\ \rho_t(0) = \alpha, \quad \rho_t(1) = \beta, \quad t \in (0,T], \end{cases} \tag{3.2.1}$$

if the following three conditions hold:

1. $\rho \in L^2(0,T;\mathscr{H}^1)$ *if* $\hat{\sigma} > 0$,
$\int_0^T \int_0^1 \Big\{ \frac{(\alpha - \rho_t(q))^2}{q^{\gamma+1}} + \frac{(\beta - \rho_t(q))^2}{(1-q)^{\gamma+1}} \Big\} \, dq \, dt < \infty$ *if* $\hat{\kappa} > 0$,

2. *ρ satisfies the weak formulation:*

$$\begin{aligned} F_{RD} := & \int_0^1 \rho_t(q) G_t(q) \, dq - \int_0^1 g(q) G_0(q) \, dq \\ & - \int_0^t \int_0^1 \rho_s(q) \Big(\frac{\hat{\sigma}^2}{2} \Delta + \partial_s \Big) G_s(q) \, dq \, ds \\ & - \hat{\kappa} \int_0^t \int_0^1 G_s(q) \Big(\frac{\alpha - \rho_s(q)}{q^{\gamma+1}} + \frac{\beta - \rho_s(q)}{(1-q)^{\gamma+1}} \Big) \, dq \, ds = 0, \end{aligned} \tag{3.2.2}$$

for all $t \in [0,T]$ and any function $G \in C_c^{1,2}([0,T] \times [0,1])$,

3. *if $\hat{\sigma} > 0$ then $\rho_t(0) = \alpha, \quad \rho_t(1) = \beta$ for all $t \in [0,T]$.*

Remark 3.2.2. *Observe that in the case $\hat{\sigma} > 0$ and $\hat{\kappa} = 0$ we recover the heat equation with Dirichlet boundary conditions. If $\hat{\sigma} = 0$ the equation does not have the diffusion term.*

Definition 3.2.3. *Let $\hat{\sigma} > 0$ and $\hat{m} \geq 0$ be some parameters. We say that $\rho : [0,T] \times [0,1] \to [0,1]$ is a weak solution of the heat equation with Robin boundary conditions*

$$\begin{cases} \partial_t \rho_t(q) = \frac{\hat{\sigma}^2}{2} \Delta \rho_t(q), \quad (t,q) \in [0,T] \times (0,1), \\ \partial_q \rho_t(0) = \frac{2\hat{m}}{\hat{\sigma}^2}(\rho_t(0) - \alpha), \quad \partial_q \rho_t(1) = \frac{2\hat{m}}{\hat{\sigma}^2}(\beta - \rho_t(1)), \quad t \in [0,T], \end{cases} \tag{3.2.3}$$

if the following two conditions hold:

1. $\rho \in L^2(0,T;\mathscr{H}^1)$,

2. ρ *satisfies the weak formulation:*

$$\begin{aligned} F_{Rob} := \int_0^1 \rho_t(q) G_t(q)\, dq - \int_0^1 g(q) G_0(q)\, dq \\ - \int_0^t \int_0^1 \rho_s(q) \Big(\frac{\hat{\sigma}^2}{2} \Delta + \partial_s \Big) G_s(q)\, dq\, ds \\ + \frac{\hat{\sigma}^2}{2} \int_0^t \{ \rho_s(1) \partial_q G_s(1) - \rho_s(0) \partial_q G_s(0) \}\, ds \\ - \hat{m} \int_0^t \{ G_s(0)(\alpha - \rho_s(0)) + G_s(1)(\beta - \rho_s(1)) \}\, ds = 0, \end{aligned} \tag{3.2.4}$$

for all $t \in [0,T]$, any function $G \in C^{1,2}([0,T] \times [0,1])$.

Remark 3.2.4. *Observe that in the case $\hat{m} = 0$ the equation above is the heat equation with Neumann boundary conditions.*

3.2.2 Hydrodynamic Limit: finite variance

Recall the notion of the empirical measure given in Section 2.6 and note that in this case we have

$$\pi_t^N(\eta, dq) := \pi^N(\eta_{t\theta(N)}, dq)$$

and we note that, in this case, the time scale $\theta(N)$ will change with the range of θ, contrarily to what happens in the model of Chapter 2. As before, let $\mathbb{P}_{\mu_N}$ be the probability measure in the Skorohod space $\mathscr{D}([0,T], \Omega_N)$ induced by the Markov process $\{\eta_{t\theta(N)} : t \geq 0\}$ and the initial probability measure μ_N and we denote by $\mathbb{E}_{\mu_N}$ the expectation with respect to $\mathbb{P}_{\mu_N}$ and let $\{\mathbb{Q}_N\}_{N\geq 1}$ be the sequence of probability measures on $\mathscr{D}([0,T], \mathscr{M}^+)$ induced by the Markov process $\{\pi_t^N ; t \geq 0\}$ and by $\mathbb{P}_{\mu_N}$.

Remark 3.2.5. *We note that due to the presence of long jumps in the system, we cannot obtain information about the empirical profile nor the two point correlation function in a simple way as we did in Section 2.5. We also note that the matrix ansatz method described in Section 2.4 in this case does not give us any information about the stationary measures for this model. This study is left for a future work.*

Let $g : [0,1] \to [0,1]$ be a measurable function and let $\{\mu_N\}_{N\geq 1}$ be a sequence of probability measures in Ω_N associated to $g(\cdot)$, see (2.7.2). The first result in this chapter is stated in the following theorem (see Figure 3.1).

Theorem 3.2.6. *Let $g : [0,1] \to [0,1]$ be a measurable function and let $\{\mu_N\}_{N\geq 1}$ be a sequence of probability measures in Ω_N associated to $g(\cdot)$. Then, for any $0 \leq t \leq T$,*

$$\lim_{N\to\infty} \mathbb{P}_{\mu_N}\Big(\eta_\cdot : \Big| \frac{1}{N-1} \sum_{x\in\Lambda_N} G\left(\tfrac{x}{N}\right) \eta_{t\theta(N)}(x) - \int_0^1 G(q)\rho_t(q)dq \Big| > \delta\Big) = 0,$$

where the time scale is given by

$$\Theta(N) = \begin{cases} N^2, & \text{if } \theta \geq 1-\gamma, \\ N^{\gamma+\theta+1}, & \text{if } \theta < 1-\gamma, \end{cases} \tag{3.2.5}$$

and $\rho_t(\cdot)$ is the unique weak solution of :

- (3.2.1) *with $\hat{\sigma} = 0$ and $\hat{\kappa} = \kappa c_\gamma$, if $\theta < 1-\gamma$;*
- (3.2.1) *with $\hat{\sigma} = \sigma$ and $\hat{\kappa} = \kappa c_\gamma$, if $\theta = 1-\gamma$;*
- (3.2.1) *with $\hat{\sigma} = \sigma$ and $\hat{\kappa} = 0$, if $\theta \in (1-\gamma, 1)$;*
- *(3.2.3) with $\hat{\sigma} = \sigma$ and $\hat{m} = \frac{\kappa}{2}$, if $\theta = 1$;*
- *(3.2.3) with $\hat{\sigma} = \sigma$ and $\hat{m} = 0$, if $\theta > 1$.*

Remark 3.2.7. *We note that for a probability transition $p(\cdot)$ which is symmetric and with finite variance the last three regimes obtained above are in force (however* (3.2.1) *with $\hat{\kappa} = 0$ is obtained for $\theta \in [0,1)$). We note that the two first regimes depend on the specific choice of the transition probability $p(\cdot)$ that we have assumed in* (3.1.1). *We also note that if we impose that the higher moments of $p(\cdot)$ are finite then the regime* (3.2.1) *with $\hat{\kappa} = 0$ can be reached for $\theta \in [v, 1)$ where $v < 0$ depends on the finiteness of the moments of $p(\cdot)$.*

Remark 3.2.8. *We note that the solution of the hydrodynamic equation depends on the parameter κ which appears at the boundary dynamics in two different regimes of θ, namely $\theta = 1-\gamma$ and $\theta = 1$.*

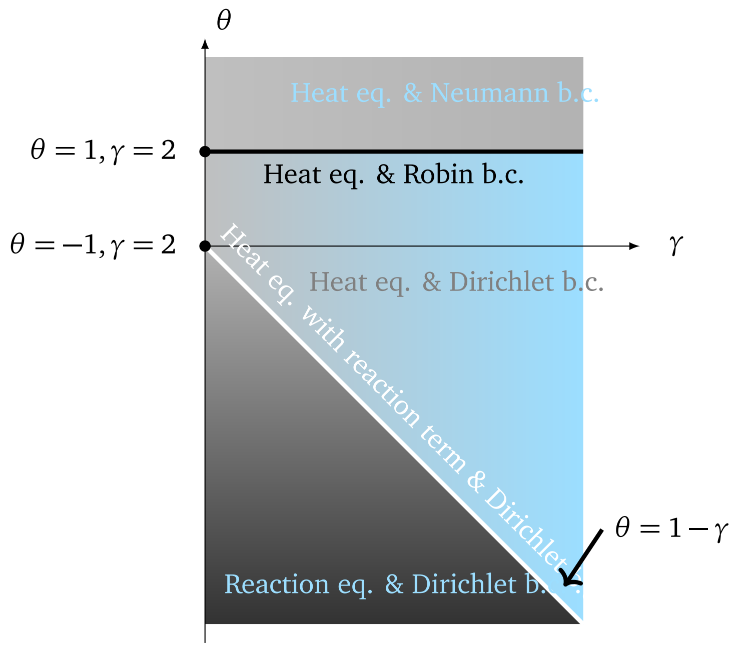


Figure 3.1: The five different hydrodynamic regimes in terms of γ and θ.

Now note that as before, the stationary solutions of the hydrodynamic limits in the case $\theta > 1-\gamma$ are standard and for that reason they are ommited. On the other hand, the form and properties of the stationary solutions in the case $\theta \leq 1-\gamma$ are more complicated to obtain in the case $\theta = 1-\gamma$. This problem is studied in more details in [15] for a slighlty different dynamics. Here we only present some graphs of the stationary solutions and refer the interested reader to [15] for a complete description on the behavior of those solutions. Below we draw the graph of these stationary solutions for a choice of $\alpha = 0.2$ and $\beta = 0.8$.

The proof of Theorem 3.2.6 is described in Section 2.7 below Figure 2.4 and for that reason many steps now are omitted. The proof of tightness of the sequence $\{\mathbb{Q}_N\}_{N\geq 1}$ is quite similar to the one given in Section 2.9. The characterization of limit points is also close to the one given in Section 2.10, the only difference comes at the level of the identification of the density as a weak solution of the corresponding partial differential equation. For that purpose, the next section is dedicated to the presentation of an heuristic argument to deduce the weak formulation for the solution of the corresponding hydrodynamic equation. The adaptation of the rest of the arguments to this new dynamics is

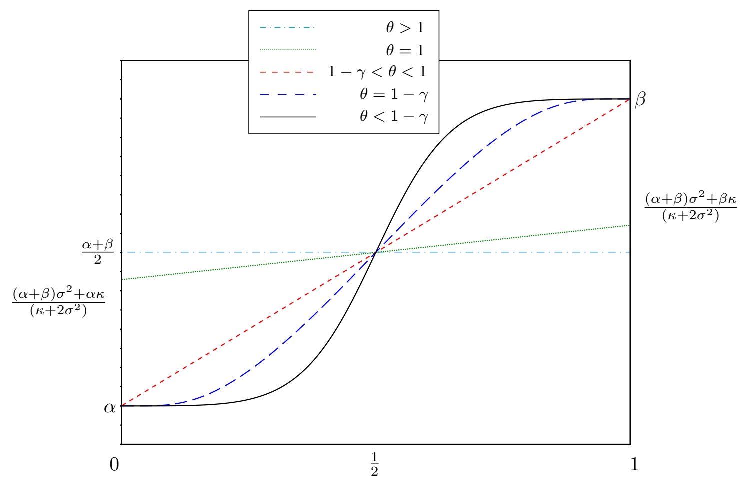


Figure 3.2: Stationary solutions of the hydrodynamic equations.

left to the reader.

3.2.3 Heuristics for hydrodynamic equations: finite variance

As in Section 2.8, the identification of the density $\rho_t(\cdot)$ as a weak solution of the corresponding hydrodynamic equation is obtained by using auxiliary martingales. Fix then a function $G : [0,1] \to \mathbb{R}$ which does not depend on time and which is two times continuously differentiable. As in Section 2.8, we use Dynkin's formula and we note that

$$\int_0^t \Theta(N)\mathscr{L}_N(\langle \pi_s^N, G\rangle)\,ds = \frac{\Theta(N)}{N-1}\int_0^t \sum_{x\in\Lambda_N} \tilde{\mathscr{L}}_N G(\tfrac{x}{N})\eta_{s\theta(N)}(x)\,ds$$
$$+\frac{\kappa\Theta(N)}{(N-1)N^\theta}\int_0^t \sum_{y\in\{0,N\}}\sum_{x\in\Lambda_N} G(\tfrac{x}{N})p(y-x)(r(y)-\eta_{s\theta(N)}(x))\,ds, \tag{3.2.6}$$

where for all $x\in\Lambda_N$

$$(\tilde{\mathscr{L}}_N G)(\tfrac{x}{N}) = \sum_{y\in\Lambda_N} p(y-x)\left[G(\tfrac{y}{N})-G(\tfrac{x}{N})\right]. \tag{3.2.7}$$

Now, we extend the first sum in (3.2.6) to all the integers so that we extend the function G to $\mathbb{R}$ in such a way that it remains two times continuously differentiable. By the definition of $\tilde{\mathscr{L}}_N$, we get that

$$\begin{aligned}\frac{\Theta(N)}{N-1}\int_0^t \sum_{x\in\Lambda_N}\tilde{\mathscr{L}}_N G(\tfrac{x}{N})\eta_{s\theta(N)}(x)\,ds\\ =&\frac{\Theta(N)}{N-1}\int_0^t \sum_{x\in\Lambda_N}(K_N G)(\tfrac{x}{N})\eta_{s\theta(N)}(x)\,ds\\ -&\frac{\Theta(N)}{N-1}\int_0^t \sum_{x\in\Lambda_N}\sum_{y\le 0}\big[G(\tfrac{y}{N})-G(\tfrac{x}{N})\big]p(x-y)\eta_{s\theta(N)}(x)\,ds\\ -&\frac{\Theta(N)}{N-1}\int_0^t \sum_{x\in\Lambda_N}\sum_{y\ge N}\big[G(\tfrac{y}{N})-G(\tfrac{x}{N})\big]p(x-y)\eta_{s\theta(N)}(x)\,ds,\end{aligned} \tag{3.2.8}$$

where

$$(K_N G)(\tfrac{x}{N})=\sum_{y\in\mathbb{Z}}p(y-x)\big[G(\tfrac{y}{N})-G(\tfrac{x}{N})\big]. \tag{3.2.9}$$

Now, we are going to analyse how the different boundary conditions appear on the hydrodynamic equations given in Section 3.2.1 from this dynamics.

.The case $\theta < 1-\gamma$

Take a function $G:(0,1)\to\mathbb{R}$ two times continuously differentiable and with compact support in $(0,1)$, so that we can choose an extension by 0 outside of the support of G. Since $\Theta(N)=N^{\gamma+\theta+1}$ (see the statement of Theorem 3.2.6) a simple computation shows that the first term in (3.2.8) vanishes for $\theta<1-\gamma$. Indeed, by a Taylor expansion on G and the fact that $p(\cdot)$ is mean zero, we have that

$$N^{\gamma+\theta+1}\sum_{y\in\mathbb{Z}}(G(\tfrac{y+x}{N})-G(\tfrac{x}{N}))p(y)$$

is of same order as

$$N^{\gamma+\theta-1}G''(\tfrac{x}{N})\sum_{y\in\mathbb{Z}}y^2p(y)$$

and since $\theta<1-\gamma$ last expression vanishes as $N\to\infty$.

Now, the second and third terms in (3.2.8) vanish as $N\to\infty$, since $\Theta(N)=N^{\gamma+\theta+1}$ and $\theta<1-\gamma$. Note that since G vanishes outside $(0,1)$, those terms can be rewritten as

$$\frac{\Theta(N)}{N-1}\int_0^t \sum_{x\in\Lambda_N} G(\tfrac{x}{N})r_N^-(\tfrac{x}{N})\eta_{s\theta(N)}(x)\,ds + \frac{\Theta(N)}{N-1}\int_0^t \sum_{x\in\Lambda_N} G(\tfrac{x}{N})r_N^+(\tfrac{x}{N})\eta_{s\theta(N)}(x)\,ds, \tag{3.2.10}$$

where

$$r_N^-(\tfrac{x}{N}) = \sum_{y\geq x} p(y), \quad r_N^+(\tfrac{x}{N}) = \sum_{y\leq x-N} p(y). \tag{3.2.11}$$

We observe that, for any $a\in(0,1)$, uniformly in $u\in(a,1-a)$, as $N\to\infty$:

$$N^\gamma r_N^-([uN]) \to c_\gamma \gamma^{-1}u^{-\gamma} := r^-(u), \quad N^\gamma r_N^+([uN]) \to c_\gamma\gamma^{-1}(1-u)^{-\gamma} := r^+(u). \tag{3.2.12}$$

Now we note that we can bound from above, for example the term at the left hand side in (3.2.10) by $N^{\theta+1}$ times

$$\frac{1}{N-1}\int_0^t \sum_{x\in\Lambda_N} N^\gamma r_N^-(\tfrac{x}{N})\,|G(\tfrac{x}{N})|$$

because $|\eta_{sN^{\gamma+\theta}}(x)|\leq 1$ for all $s>0$. Since $\theta<-1$ and since the previous sum converges to the (finite) integral of $|G|r^-$ on $(0,1)$, by (3.2.12), the previous display vanishes as $N\to\infty$. Now we look at the boundary terms in (3.2.6), which can be written, for the choice of $\Theta(N)=N^{\gamma+\theta+1}$, as:

$$\frac{\kappa N^{\gamma+1}}{N-1}\int_0^t \sum_{y\in\{0,N\}}\sum_{x\in\Lambda_N} G\big(\tfrac{x}{N}\big)p(y-x)(r(y)-\eta_{sN^{\gamma+\theta}}(x))\,ds$$

which is equal to

$$\kappa\int_0^t \langle \alpha-\pi_s^N, Gp\rangle + \langle \beta-\pi_s^N, G\tilde{p}\rangle\,ds,$$

where $\tilde{p}(q)=p(1-q)$, and can be replaced, thanks to the fact that G has compact support, by

$$\kappa\int_0^1 G(q)\Big(p(q)(\alpha-\rho_s(q))+\tilde{p}(q)(\beta-\rho_s(q))\Big)dq$$

as $N\to\infty$. The last convergence holds because G has compact support included in $(0,1)$ so that Gp and $G\tilde{p}$ are continuous function. From the previous computations we recognize the terms in (3.2.2) with $\hat{\kappa}=\kappa c_\gamma$ and $\hat{\sigma}=0$.

.The case $\theta = 1-\gamma$

In this case we also take a function $G:(0,1)\to\mathbb{R}$ two times continuously differentiable and with compact support in $(0,1)$, so that we can choose an extension by 0 outside of its support. In this case, since $\Theta(N)=N^2$, by Lemma 3.2.10, which we state below, the first term in (3.2.8) can be replaced, for N sufficiently big, by

$$\frac{1}{N-1}\int_0^t \sum_{x\in\Lambda_N} \tfrac{\sigma^2}{2}\Delta G(\tfrac{x}{N})\,\eta_{sN^2}(x)\,ds = \int_0^t \langle \pi_s^N, \tfrac{\sigma^2}{2}\Delta G\rangle\,ds.$$

Moreover, a similar computation to the one above shows that the second and third terms in (3.2.8) vanish as $N\to\infty$ (recall that $\Theta(N)=N^2$ and $\gamma>2$). Finally, the second term in (3.2.6) can be rewritten as

$$\frac{\kappa N^{\gamma+1}}{(N-1)}\int_0^t \sum_{y\in\{0,N\}}\sum_{x\in\Lambda_N} G(\tfrac{x}{N})p(y-x)\,(r(y)-\eta_{sN^2}(x))\,ds$$

and repeating the analysis we did in the previous case it converges, as $N\to\infty$ to

$$\kappa\int_0^t\int_0^1 G(q)\Big(p(q)(\alpha-\rho_s(q))+\tilde{p}(q)(\beta-\rho_s(q))\Big)dq\,ds.$$

As above, from the previous computations we recognize the terms in (3.2.2) with $\hat{\kappa}=\kappa c_\gamma$ and $\hat{\sigma}=\sigma$.

.The case $\theta\in(1-\gamma,1)$

Take again a function $G:(0,1)\to\mathbb{R}$ two times continuously differentiable and with compact support in $(0,1)$ and extend it by 0 outside $(0,1)$. As above, since $\Theta(N)=N^2$, by Lemma 3.2.10, which we prove below, the first term in (3.2.8) can be replaced, for N sufficiently big, by

$$\int_0^t \langle \pi_s^N, \tfrac{\sigma^2}{2}\Delta G\rangle\,ds.$$

Now, the second term in (3.2.3) equals to

$$\frac{\kappa N^{2-\theta}}{N-1}\int_0^t \sum_{y\in\{0,N\}}\sum_{x\in\Lambda_N} G(\tfrac{x}{N})p(y-x)(r(y)-\eta_{sN^2}(x))\,ds$$

and vanishes as $N \to \infty$ since $\theta > 1-\gamma$. Now, the last two terms in (3.2.8) also vanish because, for example, the second term in (3.2.8) can be written as

$$\int_0^t \frac{N^2}{N-1} \sum_{x \in \Lambda_N} G(\tfrac{x}{N}) r_N^-(\tfrac{x}{N}) \eta_{sN^2}(x)\, ds$$

which can be bounded from above by a constant times $tN^{2-\gamma}$ times a sum converging to the integral of $|G|r^-$ on $(0,1)$, and since $\gamma > 2$ this term vanishes. From this, we see the terms in (3.2.2) with $\hat{\kappa} = 0$ and $\hat{\sigma} = \sigma$.

Remark 3.2.9. *We remark here that in the last three cases, similarly to what we have seen in the case $\theta < 0$ for the models of Chapter 2 (see Remark 2.8.2), there is an extra condition in the definition of the weak solution of* (3.2.1). *In this notion of solution we need to show that the value of the profile $\rho_t(\cdot)$ is fixed at the boundary. This issue is analysed in Section A.4.*

.The case $\theta = 1$

In this case we consider a function $G : [0,1] \to \mathbb{R}$ which is two times continuously differentiable and we extend it on $\mathbb{R}$ in a two times continuously differentiable function with compact support which strictly contains $[0,1]$. Note that in this case G can take non-zero values at 0 and 1. As above, since $\Theta(N) = N^2$, by Lemma 3.2.10, which we state below and which holds for this new space of test functions, the first term in (3.2.8) can be replaced, for N sufficiently big, by

$$\int_0^t \langle \pi_s^N, \tfrac{\sigma^2}{2} \Delta G \rangle \, ds.$$

Now we look at the terms coming from the boundary, namely the last term in (3.2.6). Then, in the term for $y = 0$ of (3.2.6)(resp. for $y = N$) we do at first a Taylor expansion on G and then we replace $\eta(x)$ by the average $\overrightarrow{\eta}^{\varepsilon N}(1) = \frac{1}{\varepsilon N} \sum_{x=1}^{1+\varepsilon N} \eta(x)$ (resp. $\eta(x)$ by $\overleftarrow{\eta}^{\varepsilon N}(N-1) = \frac{1}{\varepsilon N} \sum_{x=N-1-\varepsilon N}^{N-1} \eta(x)$), which can be done as a consequence of Lemma A.3.2 as pointed out in Remark A.3.3. Moreover, note that for $y = 0$ and $y = N$ it holds that

$$\sum_{x \in \Lambda_N} p(y-x) \xrightarrow[N \uparrow \infty]{} \frac{1}{2}. \tag{3.2.13}$$

Therefore, we can write the last term in (3.2.6) as

$$\frac{\kappa}{2} \int_0^t \{(\alpha - \overrightarrow{\eta}_{sN^2}^{\varepsilon N}(1))G(0) + (\beta - \overleftarrow{\eta}_{sN^2}^{\varepsilon N}(N-1))G(1)\}\, ds,$$

plus terms that vanish as $N \to +\infty$. Since $\overrightarrow{\eta}^{\varepsilon N}_{sN^2}(1) \sim \rho_s(0)$ and $\overleftarrow{\eta}^{\varepsilon N}_{sN^2}(N-1) \sim \rho_s(1)$ last term writes as

$$\frac{\kappa}{2}\int_0^t \{(\alpha-\rho_s(0))G(0)+(\beta-\rho_s(1))G(1)\}\,ds. \tag{3.2.14}$$

Now, we analyse the two last terms in (3.2.8). Since the function G has been extended into a two times continuously differentiable function on $\mathbb{R}$, by a Taylor expansion on G we can write those terms as

$$\frac{N}{N-1}\int_0^t \sum_{x\in\Lambda_N} G'(\tfrac{x}{N})\Theta_x^-\eta_{sN^2}(x)\,ds - \frac{N}{N-1}\int_0^t \sum_{x\in\Lambda_N} G'(\tfrac{x}{N})\Theta_x^+\eta_{sN^2}(x)\,ds \tag{3.2.15}$$

plus terms that vanish as $N \to +\infty$. Above for $x \in \Lambda_N$,

$$\Theta_x^- = \sum_{y\le 0}(x-y)p(x-y) \quad \text{and} \quad \Theta_x^+ = \sum_{y\ge N}(y-x)p(x-y).$$

Note that

$$\frac{1}{N}\sum_{x\in\Lambda_N} x\Theta_x^- \xrightarrow[N\to\infty]{} 0 \quad \text{and} \quad \frac{1}{N}\sum_{x\in\Lambda_N} x\Theta_x^+ \xrightarrow[N\to\infty]{} 0. \tag{3.2.16}$$

Moreover, note that

$$\begin{aligned} \sum_{x\in\Lambda_N}\Theta_x^- &= \sum_{x\in\Lambda_N}\sum_{y\ge x} yp(y) \xrightarrow[N\uparrow\infty]{} \tfrac{\sigma^2}{2}, \\ \sum_{x\in\Lambda_N}\Theta_x^+ &= \sum_{x\in\Lambda_N}\sum_{y\ge N-x} yp(y) \xrightarrow[N\uparrow\infty]{} \tfrac{\sigma^2}{2}. \end{aligned} \tag{3.2.17}$$

In order to prove the convergence of $\sum_{x\in\Lambda_N}\Theta_x^-$ (or of $\sum_{x\in\Lambda_N}\Theta_x^+$ in (3.2.17)) we use Fubini's theorem to get that

$$\begin{aligned} \sum_{x\in\Lambda_N}\Theta_x^- &= \sum_{y\in\Lambda_N}\sum_{x=1}^{y} yp(y) + \sum_{y\ge N}\sum_{x\in\Lambda_N} yp(y) \\ &= \sum_{y\in\Lambda_N} y^2p(y) + (N-1)\sum_{y\ge N} yp(y), \end{aligned}$$

and since $\gamma > 2$ the result follows. By another Taylor expansion on G we can write (3.2.15) as

$$\frac{N}{N-1}G'(0)\int_0^t \sum_{x\in\Lambda_N}\Theta_x^-\eta_{sN^2}(x)\,ds - \frac{N}{N-1}G'(1)\int_s^t \sum_{x\in\Lambda_N}\Theta_x^+\eta_{sN^2}(x)\,ds \tag{3.2.18}$$

plus terms that vanish as $N \to +\infty$. From Lemma A.3.2 we can replace in the term on the left (resp. right) hand side of last expression $\eta_{sN^2}(x)$ by $\overrightarrow{\eta}^{\varepsilon N}_{sN^2}(1)$ (resp. $\overleftarrow{\eta}^{\varepsilon N}_{sN^2}(N-1)$). Therefore, (3.2.18) can be replaced, for N sufficiently big and for ε sufficiently small, by

$$\int_0^t G'(0)\tfrac{\sigma^2}{2}\overrightarrow{\eta}^{\varepsilon N}_{sN^2}(1) - G'(1)\tfrac{\sigma^2}{2}\overleftarrow{\eta}^{\varepsilon N}_{sN^2}(N-1)\,ds.$$

Since $\overrightarrow{\eta}^{\varepsilon N}_{sN^2}(1) \sim \rho_s(0)$ and $\overleftarrow{\eta}^{\varepsilon N}_{sN^2}(N-1) \sim \rho_s(1)$, last term tends to

$$\int_0^t G'(0)\tfrac{\sigma^2}{2}\rho_s(0) - G'(1)\tfrac{\sigma^2}{2}\rho_s(1)\,ds, \tag{3.2.19}$$

as $N \to \infty$.

Putting together (3.2.14) and (3.2.19) we see the boundary terms that appear at the right hand side of (3.2.4).

.The case $\theta > 1$

In this case we consider an arbitrary function $G : [0,1] \to \mathbb{R}$ which is two times continuously differentiable and we extend it on $\mathbb{R}$ in a two times continuously differentiable function with compact support. Its support strictly contains $[0,1]$ since G can take non-zero values at 0 and 1. As in the last case, since $\Theta(N) = N^2$, by Lemma 3.2.10, the first term in (3.2.8) can be replaced, for N sufficiently big, by

$$\int_0^t \langle \pi^N_s, \tfrac{\sigma^2}{2}\Delta G\rangle\,ds.$$

The last term in (3.2.6) vanishes, as $N \to \infty$ since, we can bound it by a constant times

$$N^{1-\theta}\sum_{x\in\Lambda_N} p(x).$$

Since $\gamma > 2$ last display vanishes if $\theta > 1$, as $N \to +\infty$. Thus, we only need to look at the expression (3.2.8). Therefore, in order to see the boundary terms that appear in (3.2.4), we can use exactly the computations already done in the case $\theta = 1$ from which we obtain (3.2.19).

We finish this section with the statement of the lemma which is used above in order to obtain the diffusion term in the equations above in the cases $\theta \geq 1-\gamma$. Its proof can be seen in [2].

Lemma 3.2.10. *Let $G:\mathbb{R}\to\mathbb{R}$ be a two times continuously differentiable function with compact support. We have*

$$\limsup_{N\to\infty}\sup_{x\in\Lambda_N}\left|N^2\sum_{y\in\mathbb{Z}}(G(\tfrac{y+x}{N})-G(\tfrac{x}{N}))p(y)-\frac{\sigma^2}{2}\Delta G(\tfrac{x}{N})\right|=0.$$

3.3 The infinite variance case

In this section we analyse the case in which $p(\cdot)$ is as in (3.1.1) but now $\gamma\in(1,2)$ so that $p(\cdot)$ has mean zero but infinite variance. We also consider only the case where $\theta=-1$, but we note that in the regime $\theta<-1$ the behavior of the system, when we take the time scale $\Theta(N)=N^{\gamma+\theta+1}$ is the same as when $\theta<1-\gamma$ and when $p(\cdot)$ has finite variance, that is, it is given by the weak solution of (3.2.1) with $\hat{\sigma}=0$ and $\hat{\kappa}=\kappa c_\gamma$. The other regimes are open and seem to be quite challenging. Recall the infinitesimal generator given in (3.1.2) and (3.1.3) and since we are restricted to the case $\theta=-1$, we consider the Markov process speeded up in the time scale $\Theta(N)=N^\gamma$, so that $\{\eta_{tN^\gamma}\,:\,t\geq 0\}$ has infinitesimal generator given by $N^\gamma\mathscr{L}_N$. As in Section 2.4 we can prove that the Bernoulli product measures ν_ρ^N as defined in (2.4.1) are reversible when we consider $\alpha=\beta=\rho$. The proof is quite similar to the one given in Lemma 2.4.1 and for that reason it is omitted.

3.3.1 Hydrodynamic equations: infinite variance

We can now give the definition of the weak solution of the hydrodynamic equation that will be derived in this section when $p(\cdot)$ is assumed to have infinite variance.

Recall the notations introduced in the beginning of Section 2.6. The fractional Laplacian operator of exponent $\gamma/2$ denoted by $(-\Delta)^{\gamma/2}$ is defined on the set of functions $G:\mathbb{R}\to\mathbb{R}$ such that

$$\int_{-\infty}^{\infty}\frac{|G(q)|}{(1+|q|)^{1+\gamma}}dq<\infty \tag{3.3.1}$$

by

$$(-\Delta)^{\gamma/2}G(q)=c_\gamma\lim_{\varepsilon\to 0}\int_{-\infty}^{\infty}\mathbf{1}_{|q-v|\geq\varepsilon}\,\frac{G(q)-G(v)}{|q-v|^{1+\gamma}}dv \tag{3.3.2}$$

provided the limit exists, which is the case, for example, if G is in the Schwartz space $\mathbb{S}(\mathbb{R})$ and where c_γ is fixed in (3.1.1). Up to a multiplicative constant, $-(-\Delta)^{\gamma/2}$ is the generator of a γ-Lévy stable process.

We define the operator $\mathbb{L}$ by its action on functions $G \in C_c^\infty((0,1))$, by

$$\forall q \in (0,1), \quad (\mathbb{L}G)(q) = c_\gamma \lim_{\varepsilon \to 0} \int_0^1 \mathbf{1}_{|q-v|\geq \varepsilon} \frac{G(v)-G(q)}{|q-v|^{1+\gamma}} dv.$$

The operator $\mathbb{L}$ is called the *regional fractional Laplacian* on $(0,1)$. The semi inner-product $\langle \cdot, \cdot \rangle_{\gamma/2}$ is defined on the set $C_c^\infty((0,1))$ by

$$\langle G, H \rangle_{\gamma/2} = \frac{c_\gamma}{2} \iint_{[0,1]^2} \frac{(H(q)-H(v))(G(q)-G(v))}{|q-v|^{1+\gamma}} dq dv. \tag{3.3.3}$$

The corresponding semi-norm is denoted by $\|\cdot\|_{\gamma/2}$. Observe that for any $G, H \in C_c^\infty((0,1))$ we have that

$$-\int_0^1 G(q) \mathbb{L} H(q)\, dq = -\int_0^1 \mathbb{L} G(q) H(q)\, dq = \langle G, H \rangle_{\gamma/2}$$

and note that for all $q \in (0,1)$,

$$(\mathbb{L}G)(q) = -(-\Delta)^{\gamma/2} G(q) + V_1(q) G(q) \tag{3.3.4}$$

where $V_1(q) = r^-(q) + r^+(q)$, see (3.2.12), that is, $V_1(\cdot)$ is given on $q \in (0,1)$ by:

$$V_1(q) = c_\gamma \gamma^{-1} \Big(\frac{1}{q^\gamma} + \frac{1}{(1-q)^\gamma} \Big). \tag{3.3.5}$$

Definition 3.3.1. *The Sobolev space $\mathscr{H}^{\gamma/2}$ consists of all square integrable functions $g : (0,1) \to \mathbb{R}$ such that $\|g\|_{\gamma/2} < \infty$. This is a Hilbert space for the norm $\|\cdot\|_{\mathscr{H}^{\gamma/2}}$ defined by*

$$\|g\|^2_{\mathscr{H}^{\gamma/2}} := \|g\|^2 + \|g\|^2_{\gamma/2}.$$

Its elements elements coincide a.e. with continuous functions. The space $L^2(0,T;\mathscr{H}^{\gamma/2})$ is the set of measurable functions $f : [0,T] \to \mathscr{H}^{\gamma/2}$ such that

$$\int_0^T \|f_t\|^2_{\mathscr{H}^{\gamma/2}} dt < \infty.$$

We now extend the definition of the regional fractional Laplacian on $(0,1)$ to the space $\mathscr{H}^{\gamma/2}$.

Definition 3.3.2. *For $\rho \in \mathscr{H}^{\gamma/2}$ we define the distribution $\mathbb{L}\rho$ by*

$$\int_0 \mathbb{L}\rho(u) G(u)\, du = \int_0^1 \rho(u) \mathbb{L} G(u)\, du, \quad G \in C_c^\infty((0,1)).$$

Let $\mathbb{L}_\kappa$ be the regional fractional Laplacian on $[0,1]$ with zero Dirichlet boundary conditions, indexed by κ, and taking the form

$$\mathbb{L}_\kappa = \mathbb{L} - \kappa \tilde{V}_1, \tag{3.3.6}$$

where

$$\tilde{V}_1(q) = p(q) + \tilde{p}(q) = c_\gamma \Big(\frac{1}{q^{\gamma+1}} + \frac{1}{(1-q)^{\gamma+1}} \Big). \tag{3.3.7}$$

Above $\tilde{p}(q) = p(1-q)$. Below $g : [0,1] \to [0,1]$ is a measurable function and it is the initial condition of the partial differential equation that we obtain in this section.

Definition 3.3.3. *Let $\kappa > 0$ be some parameter. We say that $\rho^\kappa : [0,T] \times [0,1] \to [0,1]$ is a weak solution of the regional fractional reaction-diffusion equation with Dirichlet boundary conditions given by*

$$\begin{cases} \partial_t \rho_t^\kappa(q) = \mathbb{L}_\kappa \rho_t^\kappa(q) + \kappa \tilde{V}_0(q), & (t,q) \in [0,T] \times (0,1), \\ \rho_t^\kappa(0) = \alpha, \quad \rho_t^\kappa(1) = \beta, \quad t \in [0,T], \end{cases} \tag{3.3.8}$$

where

$$\tilde{V}_0(q) = \alpha p(q) + \beta \tilde{p}(q) = c_\gamma \Big(\frac{\alpha}{q^{1+\gamma}} + \frac{\beta}{(1-q)^{1+\gamma}} \Big),$$

if :

i) $\rho^\kappa \in L^2(0,T; \mathscr{H}^{\gamma/2})$.

ii) $\int_0^T \int_0^1 \Big\{ \frac{(\alpha - \rho_t^\kappa(q))^2}{q^{1+\gamma}} + \frac{(\beta - \rho_t^\kappa(q))^2}{(1-q)^{1+\gamma}} \Big\} \, dq \, dt < \infty$.

iii) For all $t \in [0,T]$ and all functions $G \in C_c^{1,\infty}([0,T] \times (0,1))$ we have that

$$\begin{aligned} F_{Dir}^\kappa := &\int_0^1 \rho_t^\kappa(q) G_t(q) \, dq - \int_0^1 g(q) G_0(q) \, dq \\ &- \int_0^t \int_0^1 \rho_s^\kappa(q) \Big(\partial_s + \mathbb{L}_\kappa \Big) G_s(q) \, dq ds \\ &- \kappa \int_0^t \int_0^1 G_s(q) \tilde{V}_0(q) \, dq \, ds = 0. \end{aligned} \tag{3.3.9}$$

Remark 3.3.4. *We observe that the partial differential equation above has a unique weak solution in the sense defined above. We do not include the proof of this result in these notes but we refer the interested reader to [2] for the proof of the uniqueness for a very similar equation. The same proof gives uniqueness in this case.*

3.3.2 Hydrodynamic Limit: infinite variance case

Recall the notion of the empirical measure given in Section 2.6 and note that in this case we have

$$\pi_t^N(\eta, dq) := \pi^N(\eta_{tN^\gamma}, dq)$$

since the time scale now is equal to $\theta(N) = N^\gamma$.

The second result of this chapter is stated in the following theorem.

Theorem 3.3.5. *Let $g : [0,1] \to [0,1]$ be a measurable function and let $\{\mu_N\}_{N\geq 1}$ be a sequence of probability measures in Ω_N associated to $g(\cdot)$. Then, for any $0 \leq t \leq T$,*

$$\lim_{N\to\infty} \mathbb{P}_{\mu_N}\left(\eta_\cdot : \left| \frac{1}{N-1} \sum_{x\in\Lambda_N} G\left(\tfrac{x}{N}\right) \eta_{tN^\gamma}(x) - \int_0^1 G(q)\rho_t^\kappa(q)dq \right| > \delta \right) = 0,$$

where ρ_t^κ is the unique weak solution of (3.3.8) in the sense of Definition 3.3.3.

3.3.3 Heuristics for hydrodynamic equations: infinite variance

Fix $G : [0,1] \to \mathbb{R}$ which does not depend on time and has compact support included in $(0,1)$. Recall (3.2.6) and (3.2.8) and recall that we assumed $\theta = -1$, so that (3.2.3) now writes as

$$\begin{aligned}\int_0^t N^\gamma \mathscr{L}_N(\langle \pi_s^N, G\rangle)\, ds &= \frac{N^\gamma}{N-1}\int_0^t \sum_{x\in\Lambda_N} (\tilde{\mathscr{L}}_N G)(\tfrac{x}{N})\eta_{sN^\gamma}(x) \\ &+ \frac{\kappa N^{\gamma+1}}{(N-1)}\int_0^t \sum_{y\in\{0,N\}}\sum_{x\in\Lambda_N} G(\tfrac{x}{N})p(y-x)(r(y)-\eta_{sN^\gamma}(x))\, ds.\end{aligned} \tag{3.3.10}$$

Note that the first term on the right hand side in last display is equal to

$$\int_0^t \langle \pi_s^N, \tilde{\mathscr{L}}_N G\rangle\, ds.$$

Since from Lemma 3.3 in [4], we can deduce that

$$\lim_{N\to\infty} N^\gamma(\tilde{\mathscr{L}}_N G)(q) = (\mathbb{L}G)(q) \tag{3.3.11}$$

uniformly in $[a, 1-a]$, for all functions G with compact support included in $[a, 1-a]$, that term can be replaced by

$$\int_0^t \int_0^1 (\mathbb{L}G)(q)\rho_s^\kappa(q)\, dq\, ds, \tag{3.3.12}$$

as N goes to ∞. Now, the second term on the right hand side in (3.3.10) is equal to

$$\kappa \int_0^t \langle \alpha - \pi_s^N, Gp \rangle \, ds + \kappa \int_0^t \langle \beta - \pi_s^N, G\tilde{p} \rangle \, ds$$

and converges as $N \to \infty$ to

$$\begin{aligned}
&\kappa \int_0^t \int_0^1 (\alpha - \rho_t^\kappa(q))G(q)p(q)du + \kappa \int_0^t \int_0^1 (\beta - \rho_t^\kappa(q))G(q)\tilde{p}(q)dq \\
&= -\kappa \int_0^t \int_0^1 \rho_t^\kappa(q)G(q)\tilde{V}_1(q)dq + \kappa \int_0^t \int_0^1 G(q)\tilde{V}_0(q)dq.
\end{aligned} \tag{3.3.13}$$

Putting together (3.3.12) and (3.3.13) and using (3.3.6) we recognize the corresponding terms in (3.3.9).

Remark 3.3.6. *We finish this chapter by noting that in [3] it was studied a similar dynamics to the one described above. There we considered the same bulk dynamics with long jumps given by $p(\cdot)$ with the choice* (3.1.1) *and $\gamma \in (1,2)$ but the boundary dynamics was different. In that paper instead of considering just one boundary at each end point of the bulk, it was added infinitely many reservoirs at the left and at the right of the bullk. As in the dynamics described above, particles can be injected and removed from the system at any point of the bulk by any of the reservoirs located at $y \leq 0$ or $y \geq N$. We note that in the case of this new dynamics the results obtained in [3] are similar to those presented here, except that the transitions occur for a different value of θ and for that reason, the potential that appear in the reaction diffusion equation has a different power than the one that appears in the hydrodynamic equation in [3]. It would be very interesting to analyse other types of boundary dynamics superposed to the bulk dynamics that we defined above in order to see if we can come up with new fractional reaction diffusion equations with more tricky boundary conditions than the Dirichlet boundary conditions that we obtained here. And it would be very interesting to look at the case where $\theta > -1$, the slow boundary regime, when $p(\cdot)$ is given as above in the case of infinite variance. This is a subject to pursue in the near future.*

Appendix A

Auxiliary results

In this section we establish some technical results that are needed in order to prove the hydrodynamic limit for the models discussed in the previous chapters.

A.1 Entropy bound

From now on, we suppose that $\alpha \le \beta$. Let $\rho : [0,1] \to [0,1]$ be a function such that $\alpha \le \rho(q) \le \beta$, for all $q \in [0,1]$. Let $\nu^N_{\rho(\cdot)}$ be the Bernoulli product measure on Ω_N with marginals given by

$$\nu^N_{\rho(\cdot)}\{\eta : \eta_x = 1\} = \rho\left(\tfrac{x}{N}\right). \tag{A.1.1}$$

Given two functions $f, g : \Omega_N \to \mathbb{R}$ and a probability measure μ on Ω_N, we denote here by $\langle f, g\rangle_\mu$ the scalar product between f and g in $L^2(\Omega_N, \mu)$, that is,

$$\langle f, g\rangle_\mu = \int_{\Omega_N} f(\eta)g(\eta)\, d\mu.$$

Let $H_N(\mu|\nu^N_{\rho(\cdot)})$ be the relative entropy of a probability measure μ on Ω_N with respect to the probability measure $\nu^N_{\rho(\cdot)}$ on Ω_N. We claim that there exists a constant $C_0 := C(\alpha, \beta)$, such that

$$H_N(\mu|\nu^N_{\rho(\cdot)}) \le C_0 N. \tag{A.1.2}$$

For that purpose note that, since $\nu^N_{\rho(\cdot)}$ is product we have that

$$\nu^N_{\rho(\cdot)}(\eta) = \prod_{x=1}^{N-1} \rho(\tfrac{x}{N})^{\eta(x)}(1-\rho(\tfrac{x}{N}))^{1-\eta(x)} \ge (\alpha \wedge (1-\beta))^N$$

from where we obtain that

$$\begin{aligned} H(\mu|\nu^N_{\rho(\cdot)}) &= \sum_{\eta\in\Omega_N}\mu(\eta)\log\left(\frac{\mu(\eta)}{\nu^N_{\rho(\cdot)}(\eta)}\right)\leq\sum_{\eta\in\Omega_N}\mu(\eta)\log\left(\frac{1}{\nu^N_{\rho(\cdot)}(\eta)}\right)\\ &\leq \log\left(\left[\frac{1}{\alpha\wedge(1-\beta)}\right]^N\right)\sum_{\eta\in\Omega_N}\mu(\eta)\leq N\log\left(\frac{1}{\alpha\wedge(1-\beta)}\right)\leq C_0N. \end{aligned}$$

We remark here that below when we use as reference measure the Bernoulli product measure given in (A.1.1) we have to restrict to $\alpha\neq 0$ and $\beta\neq 1$ since in last estimate the constant $C_0=-\log(\alpha\wedge(1-\beta))$. We also note that when we use the Bernoulli product measure with a constant parameter we do not need to impose that restriction.

A.2 Estimates on Dirichlet forms

In this section we consider the model described in Chapter 3 since the results for the model of Chapter 2 can be obtained easily from the ones we derive below. In any case we present some remarks along the text about the corresponding results for the model of Chapter 2.

For a probability measure μ on Ω_N, $x,y\in\Lambda_N$ and a density function $f:\Omega_N\to[0,\infty)$ with respect to μ we introduce

$$\begin{aligned} I_{x,y}(\sqrt{f},\mu) &:= \int_{\Omega_N}\left(\sqrt{f(\eta^{x,y})}-\sqrt{f(\eta)}\right)^2 d\mu,\\ I_x^{r(y)}(\sqrt{f},\mu) &:= \int_{\Omega_N} c_x(\eta;r(y))\left(\sqrt{f(\eta^x)}-\sqrt{f(\eta)}\right)^2 d\mu. \end{aligned}$$

In last identity $y\in\{0,N\}$ and $r(0)=\alpha$ and $r(N)=\beta$. We define

$$\mathscr{D}_N(\sqrt{f},\mu) \;:=\; (\mathscr{D}_{N,0}+\mathscr{D}_{N,b})(\sqrt{f},\mu)$$

where

$$\mathscr{D}_{N,0}(\sqrt{f},\mu):=\frac{1}{2}\sum_{x,y\in\Lambda_N}p(y-x)\,I_{x,y}(\sqrt{f},\mu), \tag{A.2.1}$$

$$\mathscr{D}_{N,b}(\sqrt{f},\mu):=\frac{\kappa}{N^\theta}\sum_{y\in\{0,N\}}\sum_{x\in\Lambda_N}p(y-x)\,I_x^{r(y)}(\sqrt{f},\mu). \tag{A.2.2}$$

Note that for the models of Chapter 2 the expressions above simplify to

$$\mathscr{D}_{N,0}^{NN}(\sqrt{f},\mu) := \sum_{x\in\Lambda_N} I_{x,x+1}(\sqrt{f},\mu), \tag{A.2.3}$$

$$\mathscr{D}_{N,b}^{NN}(\sqrt{f},\mu) := \frac{\kappa}{N^\theta}\Big(I_1^\alpha(\sqrt{f},\mu) + I_{N-1}^\beta(\sqrt{f},\mu)\Big). \tag{A.2.4}$$

Our first goal is to express, for the measure $\mu = \nu_{\rho(\cdot)}^N$, a relation between the Dirichlet form defined by $-\langle \mathscr{L}_N \sqrt{f}, \sqrt{f}\rangle_{\nu_{\rho(\cdot)}^N}$ and $\mathscr{D}_N(\sqrt{f}, \nu_{\rho(\cdot)}^N)$. We claim that for any positive constant B, there exists a constant $C>0$ such that

$$\begin{aligned}\frac{1}{BN}\langle \mathscr{L}_N \sqrt{f}, \sqrt{f}\rangle_{\nu_{\rho(\cdot)}^N} \le -\frac{1}{4BN}\mathscr{D}_N(\sqrt{f}, \nu_{\rho(\cdot)}^N) + \frac{C}{BN}\sum_{x,y\in\Lambda_N} p(y-x)\Big(\rho(\tfrac{x}{N}) - \rho(\tfrac{y}{N})\Big)^2 \\ + \frac{C\kappa}{BN^{1+\theta}}\sum_{y\in\{0,N\}}\sum_{x\in\Lambda_N}\Big(\rho(\tfrac{x}{N}) - r(y)\Big)^2 p(y-x).\end{aligned} \tag{A.2.5}$$

Our aim is then to choose $\rho(\cdot)$ in order to minimize the error term, i.e. the two last terms at the right hand side of the previous inequality.

Remark A.2.1.

1. *If $p(\cdot)$ has finite variance σ^2, then:*

   - *for $\rho(\cdot)$ Lipschitz and such that $\rho(0)=\alpha$ and $\rho(1)=\beta$, we get*

$$\begin{aligned}\frac{1}{BN}\langle \mathscr{L}_N \sqrt{f}, \sqrt{f}\rangle_{\nu_{\rho(\cdot)}^N} &\le -\frac{1}{4BN}\mathscr{D}_N(\sqrt{f}, \nu_{\rho(\cdot)}^N) + \frac{C}{BN^2}\sigma^2 \\ &\quad + \frac{C\kappa}{BN^{3+\theta}}\sum_{y\in\{0,N\}}\sum_{x\in\Lambda_N}\big(y-x\big)^2 p(y-x) \\ &\le -\frac{1}{4BN}\mathscr{D}_N(\sqrt{f}, \nu_{\rho(\cdot)}^N) + \frac{C}{BN^2}\sigma^2 + \frac{C\kappa}{BN^{3+\theta}}.\end{aligned} \tag{A.2.6}$$

   - *for $\rho(\cdot)$ such that $\rho(0)=\alpha$, $\rho(1)=\beta$, Hölder of parameter $\frac{\gamma}{2}$ at the boundaries and Lipschitz inside, we get*

$$\frac{1}{BN}\langle \mathscr{L}_N \sqrt{f}, \sqrt{f}\rangle_{\nu_{\rho(\cdot)}^N} \le -\frac{1}{4BN}\mathscr{D}_N(\sqrt{f}, \nu_{\rho(\cdot)}^N) + \frac{C}{BN^2}\sigma^2 + \frac{C\kappa\log(N)}{BN^{\gamma+\theta+1}}. \tag{A.2.7}$$

- *for $\rho(\cdot)$ such that $\rho(0)=\alpha$, $\rho(1)=\beta$, Hölder of parameter $\frac{1+\gamma}{2}$ at the boundaries and Lipschitz inside, we get*

$$\frac{1}{BN}\langle \mathscr{L}_N\sqrt{f},\sqrt{f}\rangle_{\nu^N_{\rho(\cdot)}} \le -\frac{1}{4BN}\mathscr{D}_N(\sqrt{f},\nu^N_{\rho(\cdot)})+\frac{C}{BN^2}\sigma^2+\frac{C\kappa}{BN^{\gamma+\theta+1}}. \tag{A.2.8}$$

- *for $\rho(\cdot)$ constant, equal to α or to β, we have*

$$\frac{1}{BN}\langle \mathscr{L}_N\sqrt{f},\sqrt{f}\rangle_{\nu^N_{\alpha}} \le -\frac{1}{4BN}\mathscr{D}_N(\sqrt{f},\nu_\alpha)+\frac{C\kappa}{BN^{\theta+1}}. \tag{A.2.9}$$

2. *If $p(\cdot)$ is such that $p(1)=p(-1)=\frac{1}{2}$, then:*

- *for $\rho(\cdot)$ Lipschitz and such that $\rho(0)=\alpha$, $\rho(1)=\beta$ and locally constant at 0 and 1, we get*

$$\frac{1}{BN}\langle \mathscr{L}_N\sqrt{f},\sqrt{f}\rangle_{\nu^N_{\rho(\cdot)}} \le -\frac{1}{4BN}\mathscr{D}^{NN}_N(\sqrt{f},\nu^N_{\rho(\cdot)})+\frac{C}{BN^2}. \tag{A.2.10}$$

Note that the choice of asking $\rho(\cdot)$ to be locally constant at 0 and 1 turns the errors coming from the boundary dynamics to vanish.

- *for $\rho(\cdot)$ constant, equal to α or to β, then we have exactly the same error as in* (A.2.9).

3. *If $p(\cdot)$ has infinite variance, then:*

- *for $\rho(\cdot)$ Lipschitz and such that $\rho(0)=\alpha$ and $\rho(1)=\beta$, we get*

$$\begin{aligned}\frac{1}{BN}\langle \mathscr{L}_N\sqrt{f},\sqrt{f}\rangle_{\nu^N_{\rho(\cdot)}} &\le -\frac{1}{4BN}\mathscr{D}_N(\sqrt{f},\nu^N_{\rho(\cdot)})\\ &\quad+\frac{C}{BN^3}\sum_{x,y\in\Lambda_N}\frac{1}{|x-y|^{\gamma-1}}\\ &\quad+\frac{C\kappa}{BN^{3+\theta}}\sum_{y\in\{0,N\}}\sum_{x\in\Lambda_N}\big(y-x\big)^2p(y-x)\\ &\le -\frac{1}{4BN}\mathscr{D}_N(\sqrt{f},\nu^N_{\rho(\cdot)})+\frac{C}{BN^\gamma}\sigma^2+\frac{C\kappa}{BN^{\gamma+\theta+1}}.\end{aligned} \tag{A.2.11}$$

In order to prove (A.2.5) we need some intermediate results. For that purpose we recall from [2] the following two lemmas.

Lemma A.2.2. *Let $T : \eta \in \Omega_N \to T(\eta) \in \Omega_N$ be a transformation in the configuration space and $c : \eta \in \Omega_N \to c(\eta)$ be a positive local function. Let f be a density with respect to a probability measure μ on Ω_N. Then, we have that*

$$\begin{aligned} &\Big\langle c(\eta)[\sqrt{f(T(\eta))} - \sqrt{f(\eta)}]\,,\ \sqrt{f(\eta)}\Big\rangle_\mu \\ &\leq -\frac{1}{4}\int c(\eta)\Big(\Big[\sqrt{f(T(\eta))}\Big] - \Big[\sqrt{f(\eta)}\Big]\Big)^2 d\mu \\ &+ \frac{1}{16}\int \frac{1}{c(\eta)}\Big[c(\eta) - c(T(\eta))\frac{\mu(T(\eta))}{\mu(\eta)}\Big]^2 \Big(\Big[\sqrt{f(T(\eta))}\Big] + \Big[\sqrt{f(\eta)}\Big]\Big)^2 d\mu. \end{aligned} \tag{A.2.12}$$

Lemma A.2.3. *There exists a constant $C := C(\rho)$ such that for any $N \geq 1$ and density f be a density with respect to $\nu^N_{\rho(\cdot)}$*

$$\sup_{x \neq y \in \Lambda_N} \int_{\Omega_N} f(\eta^{x,y})\, d\nu^N_{\rho(\cdot)}(\eta) \;\leq\; C, \qquad \sup_{x \in \Lambda_N} \int_{\Omega_N} f(\eta^{x})\, d\nu^N_{\rho(\cdot)}(\eta) \;\leq\; C.$$

A simple consequence of the previous lemmas is the next two corollaries. Recall the bulk generator $\mathscr{L}_{N,0}$ given in (3.1.3).

Corollary A.2.4. *There exists a constant $C > 0$ (independent of f and N) such that*

$$\Big\langle \mathscr{L}_{N,0}\sqrt{f}, \sqrt{f}\Big\rangle_{\nu^N_{\rho(\cdot)}} \quad\leq\quad -\frac{1}{4}\mathscr{D}_{N,0}(\sqrt{f}, \nu^N_{\rho(\cdot)}) + C\sum_{x,y \in \Lambda_N} p(y-x)\Big(\rho(\tfrac{x}{N}) - \rho(\tfrac{y}{N})\Big)^2$$

for any density f with respect to $\nu^N_{\rho(\cdot)}$.

Now we look at the generator of the boundary dynamics given in (3.1.3).

Corollary A.2.5. *Let $\theta \in \mathbb{R}$ be fixed. There exists a constant $C > 0$ (independent of f and N) such that*

$$\begin{aligned} \langle \mathscr{L}_{N,b}\sqrt{f}, \sqrt{f}\rangle_{\nu^N_{\rho(\cdot)}} &\leq -\frac{1}{4}\mathscr{D}_{N,b}(\sqrt{f}, \nu^N_{\rho(\cdot)}) \\ &\quad + \frac{C\kappa}{N^\theta}\sum_{x \in \Lambda_N}\Big(\rho(\tfrac{x}{N}) - \alpha\Big)^2 p(x) + \frac{C\kappa}{N^\theta}\sum_{x \in \Lambda_N}\Big(\rho(\tfrac{x}{N}) - \beta\Big)^2 p(N-x) \end{aligned} \tag{A.2.13}$$

for any density f with respect to $\nu^N_{\rho(\cdot)}$.

To prove the first corollary take $c \equiv 1$, $T(\eta) = \eta^{x,y}$ and note that $|\theta^{x,y}(\eta) - 1|^2 \leq C(\rho(x/N) - \rho(y/N))^2$. To prove the second corollary we take for each $y \in \{0, N\}$, $c(\eta) = c_x(\eta; r(y))$ and $T(\eta) = \eta^x$. From the two previous corollaries the claim (A.2.5) follows.

A.3 Replacement Lemmas

In this section we prove rigorously all the replacements that were mentioned along the Sections 2.8 and 3.2.3. We first recall Lemma 5.5 of [2] adapted to our situation (with just one reservoirs at each end point of the bulk).

Lemma A.3.1. *For any density f with respect to $\nu^N_{\rho(\cdot)}$, any $x \in \Lambda_N$, any $y \in \{0, N\}$ and any positive constant A_x, there exists a constant C such that*

$$\left|\langle \eta(x) - r(y), f \rangle_{\nu^N_{\rho(\cdot)}}\right| \leq \frac{C}{A_x} I^\alpha_x(\sqrt{f}, \nu^N_{\rho(\cdot)}) + CA_x + C\left|\rho(\tfrac{x}{N}) - r(y)\right|.$$

The first replacement lemma that we prove is the one that is needed for the model of Chapter 3 when $p(\cdot)$ has finite variance for the case $\theta \geq 1$.

Lemma A.3.2. *For any $t > 0$, for $\gamma > 2$ and for any $\theta \geq 1$ we have that*

$$\lim_{\varepsilon \to 0} \lim_{N \to \infty} \mathbb{E}_{\mu_N}\left[\left|\int_0^t \sum_{x \in \Lambda_N} \Theta^-_x (\eta_{sN^2}(x) - \overrightarrow{\eta}^{\varepsilon N}_{sN^2}(1))\, ds\right|\right] = 0,$$

$$\lim_{\varepsilon \to 0} \lim_{N \to \infty} \mathbb{E}_{\mu_N}\left[\left|\int_0^t \sum_{x \in \Lambda_N} \Theta^+_x (\eta_{sN^2}(x) - \overleftarrow{\eta}^{\varepsilon N}_{sN^2}(N-1))\, ds\right|\right] = 0.$$

Proof. Below C is a constant than can change from line to line. Note that since $\theta \geq 1$ we have $\theta(N) = N^2$. We present the proof for the first term, but we note that the proof for the second one is analogous. Here we take as reference measure the Bernoulli product measure with constant parameter (for example α) and we recall (A.2.9), from where we see that

$$\frac{N}{B}\langle \mathscr{L}_N \sqrt{f}, \sqrt{f}\rangle_{\nu_\alpha} \leq -\frac{N}{4B}\mathscr{D}_N(\sqrt{f}, \nu^N_\alpha) + \frac{C\kappa}{B} N^{1-\theta} \tag{A.3.1}$$

so that the error to change Dirichlet forms vanishes as $N \to \infty$ for $\theta > 1$ and for $\theta = 1$ it vanishes when $B \to +\infty$.

By the entropy and Jensen's inequalities, the first expectation in the statement of the lemma is bounded from above, for any constant $B > 0$, by

$$\frac{H(\mu_N | \nu^N_\alpha)}{BN} + \frac{1}{BN} \log \mathbb{E}_{\nu^N_\alpha}\left[e^{BN\left|\int_0^t \sum_{x \in \Lambda_N} \Theta^-_x (\eta_{sN^2}(x) - \overrightarrow{\eta}^{\varepsilon N}_{sN^2}(1))\, ds\right|}\right].$$

We can remove the absolute value inside the exponential since $e^{|x|} \leq e^{x} + e^{-x}$ and

$$\limsup_{N\to\infty} N^{-1}\log(a_N + b_N) \leq \max\left\{\limsup_{N\to\infty} N^{-1}\log(a_N), \limsup_{N\to\infty} N^{-1}\log(b_N)\right\}. \tag{A.3.2}$$

By (A.1.2), the Feynman-Kac's formula and (A.2.9), last expression can be estimated from above by $\frac{C_0}{B}$

$$\frac{C_0}{B} + t\sup_f \Big\{ \sum_{x\in\Lambda_N} \Theta_x^- \langle \eta(x) - \overrightarrow{\eta}^{\varepsilon N}(1), f\rangle_{\nu_\alpha^N} - \frac{N}{4B}\mathscr{D}_N(\sqrt{f},\nu_\alpha) + \frac{C\kappa}{B}N^{1-\theta}\Big\}, \tag{A.3.3}$$

where the supremum is carried over all the densities f with respect to ν_α^N.

Now we have to split the sum in x, depending on whether $N-1 \geq x \geq \varepsilon N$ or $x \leq \varepsilon N - 1$. We start by the first case and we have

$$\begin{aligned}\langle \eta(x) - \overrightarrow{\eta}^{\varepsilon N}(1), f\rangle_{\nu_\alpha^N} &= \frac{1}{\varepsilon N}\sum_{y=1}^{1+\varepsilon N}\int (\eta(x)-\eta(y))f(\eta)\, d\nu_\alpha^N \\ &= \frac{1}{1+\varepsilon N}\sum_{y=1}^{\varepsilon N}\sum_{z=y}^{x-1}\int (\eta(z+1)-\eta(z))f(\eta)\, d\nu_\alpha^N.\end{aligned}$$

By writing the previous term as its half plus its half and by performing in one of the terms the change of variables η into $\eta^{z,z+1}$, for which the measure ν_α^N is invariant, we write it as

$$\frac{1}{2\varepsilon N}\sum_{y=1}^{1+\varepsilon N}\sum_{z=y}^{x-1}\int (f(\eta)-f(\eta^{z,z+1}))(\eta(z+1)-\eta(z))\, d\nu_\alpha^N.$$

By using the fact that $(a-b) = (\sqrt{a}-\sqrt{b})(\sqrt{a}+\sqrt{b})$ for any $a, b \geq 0$ and since $ab \leq \dfrac{Aa^2}{2} + \dfrac{b^2}{2A}$ for all $A > 0$, we have that

$$\begin{aligned}&\sum_{x=\varepsilon N}^{N-1}\Theta_x^- \langle \eta(x) - \overrightarrow{\eta}^{\varepsilon N}(1), f\rangle_{\nu_\alpha^N} \\ &\quad\leq \frac{A}{2}\sum_{x=\varepsilon N}^{N-1}\Theta_x^- \frac{1}{2\varepsilon N}\sum_{y=1}^{1+\varepsilon N}\sum_{z=y}^{x-1}\int \Big(\sqrt{f(\eta)}-\sqrt{f(\eta^{z,z+1})}\Big)^2 d\nu_\alpha^N \\ &\quad+ \frac{1}{2A}\sum_{x=\varepsilon N}^{N-1}\Theta_x^- \frac{1}{2\varepsilon N}\sum_{y=1}^{1+\varepsilon N}\sum_{z=y}^{x-1}\int \Big(\sqrt{f(\eta)}+\sqrt{f(\eta^{z,z+1})}\Big)^2(\eta(z+1)-\eta(z))^2 d\nu_\alpha^N.\end{aligned} \tag{A.3.4}$$

By neglecting the jumps of size bigger than one, we see that

$$\sum_{z\in\Lambda_N}\int\Big(\sqrt{f(\eta)}-\sqrt{f(eta^{z,z+1})}\Big)^2\,d\nu_\alpha^N\ \le C\,\mathscr{D}_{N,0}(\sqrt{f},\nu_\alpha^N).$$

Therefore, by using also (3.2.16), the first term at the right hand side of (A.3.4) can be bounded from above by

$$\frac{A}{4}\sum_{x=\varepsilon N}^{N-1}\Theta_x^-\sum_{z\in\Lambda_N}\int\Big(\sqrt{f(\eta)}-\sqrt{f(\eta^{z,z+1})}\Big)^2\le CA\mathscr{D}_{N,0}(\sqrt{f},\nu_\alpha^N).\qquad\text{(A.3.5)}$$

Recall (A.2.9) and observe that $\mathscr{D}_N(\sqrt{f},\nu_\alpha^N)\ge\mathscr{D}_{N,0}(\sqrt{f},\nu_\alpha^N)$. Then we choose the constant A in the form $A=CN/B$ for some constant C. Moreover, for this choice of A, we can bound from above the last term at the right hand side of (A.3.4) by (use Lemma A.2.3)

$$\begin{aligned}&\frac{B}{N}\sum_{x=\varepsilon N}^{N-1}\Theta_x^-\frac{1}{2\varepsilon N}\sum_{y=1}^{\varepsilon N}\sum_{z=y}^{x-1}\int(\sqrt{f(\eta)}+\sqrt{f(\eta^{z,z+1})})^2(\eta(z+1)-\eta(z))^2d\,\nu_\alpha^N\\&\qquad\le C\frac{B}{N}\sum_{x\in\Lambda_N}x\Theta_x^-\end{aligned}\qquad\text{(A.3.6)}$$

which vanishes as $N\to\infty$ by (A.2.9). Note that the previous result holds for any $\varepsilon>0$. Now we analyse the case when $x\le\varepsilon N-1$. In that case, we write

$$\begin{aligned}\langle\eta(x)-\overrightarrow{\eta}^{\varepsilon N}(1),f\rangle_{\nu_\alpha^N}&=\frac{1}{1+\varepsilon N}\sum_{y=1}^{\varepsilon N}\int(\eta(x)-\eta(y))f(\eta)\,d\,\nu_\alpha^N\\&=\frac{1}{\varepsilon N}\sum_{y=1}^{x-1}\sum_{z=y}^{x-1}\int(\eta(z+1)-\eta(z))f(\eta)\,d\,\nu_\alpha^N\\&-\frac{1}{\varepsilon N}\sum_{y=x+1}^{1+\varepsilon N}\sum_{z=x}^{y-1}\int(\eta(z+1)-\eta(z))f(\eta)\,d\,\nu_\alpha^N.\end{aligned}$$

and the same estimates as before give that there exists a constant $C>0$ such that for any $A>0$,

$$\sum_{x=1}^{\varepsilon N-1}\Theta_x^-\langle\eta(x)-\overrightarrow{\eta}^{\varepsilon N}(1),f\rangle_{\nu_\alpha^N}\le C\left[AD_N(\sqrt{f},\nu_\alpha^N)+\frac{\varepsilon N}{A}\sum_{x=1}^{\varepsilon N-1}\Theta_x^-\right].$$

Recall (A.2.9) and (3.2.16). Then, we choose $A=N/8CB$ and the result follows. □

Remark A.3.3. *We note that above, if we change in the statement of the lemma* Θ_x^- *by* r_N^- *(resp.* Θ_x^+ *by* r_N^+*) then the same result holds by performing exactly the same estimates as above, because what we need is that*

$$\sum_{x\in\Lambda_N}\Theta_x^{\pm}<+\infty \quad \text{and} \quad \frac{1}{N}\sum_{x\in\Lambda_N}x\Theta_x^{\pm}\to 0 \tag{A.3.7}$$

which also holds for $r_N^{\pm}$ *instead of* $\Theta_x^{\pm}$ *since* $\gamma>2$.

Remark A.3.4. *Let us see now what the previous lemma says when* $p(1)=p(-1)=\frac{1}{2}$*. In this case we note that we have the same estimate as in* (A.3.1)*, see 2. in Remark A.2.1 and also note that* $\Theta_x^-\neq 0$ *for* $x=1$ *and* $\Theta_x^-=0$ *for* $x\neq 1$*. Moreover,* $\Theta_1^-=p(1)=\frac{1}{2}$*, so that the result above reads as*

$$\lim_{\varepsilon\to 0}\lim_{N\to\infty}\mathbb{E}_{\mu_N}\Big[\Big|\int_0^t(\eta_{sN^2}(1)-\overrightarrow{\eta}^{\varepsilon N}_{sN^2}(1))\,ds\Big|\Big]=0.$$

$$\lim_{\varepsilon\to 0}\lim_{N\to\infty}\mathbb{E}_{\mu_N}\Big[\Big|\int_0^t(\eta_{sN^2}(N-1)-\overleftarrow{\eta}^{\varepsilon N}_{sN^2}(N-1))\,ds\Big|\Big]=0.$$

A.4 Fixing the profile at the boundary

Let $\mathbb{Q}$ be a limit point of the sequence $\{\mathbb{Q}_N\}_{N\geq 1}$ and assume, without lost of generality, that $\{\mathbb{Q}_N\}_{N\geq 1}$ converges to $\mathbb{Q}$, as $N\to+\infty$. In this section we prove that for the model of Chapter 3 if $\theta\in[1-\gamma,1)$ (and also for the model of Chapter 2 when $\theta<0$) that the profile satisfies $\rho_t(0)=\alpha$ and $\rho_t(1)=\beta$ for $t\in(0,T]$ a.e. We present the proof for $\rho_t(0)=\alpha$ but the other case is completely analogous.

Recall (2.8.4). Observe that

$$\mathbb{E}_{\mu_N}\Big[\Big|\int_0^t(\overrightarrow{\eta}^{\varepsilon N}_{sN^2}(1)-\alpha)\,ds\Big|\Big]=\mathbb{E}_{\mathbb{Q}_N}\Big[\Big|\int_0^t(\langle\pi_s,\iota_\varepsilon^0\rangle-\alpha)\,ds\Big|\Big]$$

where $\iota_\varepsilon^0(\cdot)=\varepsilon^{-1}\mathbf{1}_{(0,\varepsilon)}(\cdot)$. Therefore we have that for any $\delta>0$,

$$\mathbb{Q}_N\Big[\Big|\int_0^t(\langle\pi_s,\iota_\varepsilon^0\rangle-\alpha)\,ds\Big|>\delta\Big]\;\leq\;\delta^{-1}\,\mathbb{E}_{\mu_N}\Big[\Big|\int_0^t(\overrightarrow{\eta}^{\varepsilon N}_{sN^2}(1)-\alpha)\,ds\Big|\Big].$$

Note that ι_ε^0 is not a continuous function so the set $\Big\{\pi\,;\,\Big|\int_0^t(\langle\pi_s,\iota_\varepsilon^0\rangle-\alpha)\,ds\Big|>\delta\Big\}$ is not an open set in the Skorohod topology, but, a simple argument as we did in

Section 2.10.3 allows to overcome the problem. Therefore, by Portemanteau's Theorem we conclude that

$$\mathbb{Q}\left[\Big|\int_0^t (\langle \pi_s, \iota_\varepsilon^0\rangle - \alpha)\,ds\Big| > \delta\right] \le \delta^{-1} \liminf_{N\to\infty} \mathbb{E}_{\mu_N}\left[\Big|\int_0^t (\overrightarrow{\eta}^{\varepsilon N}_{sN^2}(1) - \alpha)\,ds\Big|\right].$$

Now, if we are able to prove that the right hand side of the previous inequality is zero, since we have that $\mathbb{Q}$ a.s. $\pi_s(dq) = \rho_s(q)dq$ with ρ_s a continuous function in 0 for a.e. s, by taking the limit $\varepsilon \to 0$, we can deduce that $\mathbb{Q}$ a.s. $\rho_s(0) = \alpha$ for s a.e. The result follows from the next lemma.

Lemma A.4.1. *For any $t \in [0,T]$ we have that*

$$\lim_{\varepsilon\to 0}\lim_{N\to\infty} \mathbb{E}_{\mu_N}\left[\Big|\int_0^t (\overrightarrow{\eta}^{\varepsilon N}_{sN^2}(1) - \alpha)\,ds\Big|\right] = 0,$$
$$\lim_{\varepsilon\to 0}\lim_{N\to\infty} \mathbb{E}_{\mu_N}\left[\Big|\int_0^t (\overleftarrow{\eta}^{\varepsilon N}_{sN^2}(N-1) - \beta)\,ds\Big|\right] = 0.$$

To prove last lemma we use a two step procedure. First we replace, when integrated in time, $\eta_{sN^2}(1)$ by α and then we replace $\eta_{sN^2}(1)$ by $\eta^{\varepsilon N}_{sN^2}(1)$. This is the content of the next two lemmas.

Lemma A.4.2. *For $\gamma > 1$, for $1-\gamma \le \theta < 1$ and for $t \in [0,T]$ we have that*

$$\lim_{N\to\infty} \mathbb{E}_{\mu_N}\left[\Big|\int_0^t (\eta_{sN^2}(1) - \alpha)\,ds\Big|\right] = 0,$$
$$\lim_{N\to\infty} \mathbb{E}_{\mu_N}\left[\Big|\int_0^t (\eta_{sN^2}(N-1) - \beta)\,ds\Big|\right] = 0.$$

Proof. We give the proof for the first display, but we note that for the other one it is similar. Fix a Lipschitz profile $\rho(\cdot)$ such that $\alpha = \rho(0) \le \rho(\cdot) \le \rho(1) = \beta$ and $\rho(\cdot)$ is $\frac{\gamma}{2}$-Hölder at the boundary. From (A.2.7) that we know that

$$\frac{N}{B}\langle \mathscr{L}_N\sqrt{f}, \sqrt{f}\rangle_{\nu^N_{\rho(\cdot)}} \le -\frac{N}{4B}\mathscr{D}_N(\sqrt{f}, \nu^N_{\rho(\cdot)}) + \frac{C}{B}\sigma^2 + \frac{C\kappa\log(N)}{BN^{\gamma+\theta+1}}. \tag{A.4.1}$$

By the entropy inequality, for any $B > 0$, the previous expectation is bounded from above by

$$\frac{H(\mu_N|\nu^N_{\rho(\cdot)})}{BN} + \frac{1}{BN}\log \mathbb{E}_{\nu^N_{\rho(\cdot)}}\Big[e^{BN\left|\int_0^t(\eta_{sN^2}(1)-\alpha)\,ds\right|}\Big].$$

By (A.1.2), Jensen's inequality and the Feynman-Kac's formula and noting, as we did in the last proof, that we can remove the absolute value inside the exponential, last display can be estimated from above by

$$\frac{C_0}{B} + t \sup_f \left\{ \langle \eta(1) - \alpha, f \rangle_{\nu^N_{\rho(\cdot)}} - \frac{N}{4B} \mathscr{D}_N(\sqrt{f}, \nu^N_{\rho(\cdot)}) + \frac{C}{B}\sigma^2 + \frac{C\kappa}{BN^{\gamma+\theta-1}} \right\}, \tag{A.4.2}$$

where the supremum is carried over all the densities f with respect to $\nu^N_{\rho(\cdot)}$. By Lemma A.3.1, since $\rho(\cdot)$ is $\gamma/2$-Hölder at the boundaries, for any $A > 0$, the first term in the supremum in (A.4.2) is bounded from above by

$$C\left[\frac{1}{A} I_1^{\alpha}(\sqrt{f}, \nu^N_{\rho(\cdot)}) + A + \frac{1}{N^{\gamma/2}}\right]$$

for some constant $C > 0$ independent of f and A. Moreover from (A.2.7), since

$$\mathscr{D}_N(\sqrt{f}, \nu^N_{\rho(\cdot)}) \geq \mathscr{D}_{N,b}(\sqrt{f}, \nu^N_{\rho(\cdot)})$$

and $\gamma + \theta - 1 > 0$, by choosing $A = 4C(p(1))^{-1}BN^{\theta-1}$, we get then that the expression inside the brackets in (A.4.2) is bounded from above by

$$4C^2\frac{BN^{\theta-1}}{p(1)} + \frac{C}{N^{\gamma/2}} + \frac{C}{B}.$$

Now if $p(1) \neq 0$, then the proof follows by sending first $N \to \infty$ and then $B \to \infty$. For $\gamma + \theta - 1 = 0$ the same proof as above holds, the only difference is that we use a Lipschitz profile $\rho(\cdot)$ such that $\alpha = \rho(0) \leq \rho(\cdot) \leq \rho(1) = \beta$ and $\rho(\cdot)$ is $\frac{\gamma+1}{2}$-Hölder at the boundaries. From (A.2.8) that we know that

$$\frac{N}{B}\langle \mathscr{L}_N \sqrt{f}, \sqrt{f}\rangle_{\nu^N_{\rho(\cdot)}} \leq -\frac{N}{4B}\mathscr{D}_N(\sqrt{f}, \nu^N_{\rho(\cdot)}) + \frac{C}{B}\sigma^2 + \frac{C\kappa}{B}, \tag{A.4.3}$$

and with last bound and the previous argument the proof ends. □

Remark A.4.3. *The previous lemma tells us that for the model of Chapter 2 and for $\theta < 1$ and $t \in [0, T]$ we have that*

$$\lim_{N\to\infty} \mathbb{E}_{\mu_N}\left[\left|\int_0^t (\eta_{sN^2}(1) - \alpha)\, ds\right|\right] = 0,$$
$$\lim_{N\to\infty} \mathbb{E}_{\mu_N}\left[\left|\int_0^t (\eta_{sN^2}(N-1) - \beta)\, ds\right|\right] = 0.$$

Note that the previous proof follows since we have the bound (A.2.10) *and in this model $p(1) = \frac{1}{2}$.*

Remark A.4.4. *We note that for the case where $p(1) = 0$ above what we have to do is to use the two step procedure with a point z such that $p(z) \neq 0$, from where we get that:*

$$\lim_{N\to\infty} \mathbb{E}_{\mu_N}\Big[\Big|\int_0^t (\eta_{sN^2}(z)-\alpha)\,ds\Big|\Big] = 0$$

and the same result holds by changing α to β.

Now we prove the second part of the two step procedure.

Lemma A.4.5. *For $1-\gamma \leq \theta < 1$ and $t > 0$ we have that*

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\lim_{N\to\infty} \mathbb{E}_{\mu_N}\Big[\Big|\int_0^t \overrightarrow{\eta}^{\varepsilon N}_{sN^2}(1) - \eta_{sN^2}(1)\,ds\Big|\Big] = 0,\\
&\lim_{\varepsilon\to 0}\lim_{N\to\infty} \mathbb{E}_{\mu_N}\Big[\Big|\int_0^t \overleftarrow{\eta}^{\varepsilon N}_{sN^2}(N-1) - \eta_{sN^2}(N-1)\,ds\Big|\Big] = 0.
\end{aligned} \tag{A.4.4}$$

Proof. We present the proof of the first item, but we note that for the second it is exactly the same. When $\gamma+\theta-1 > 0$, we fix a Lipcshitz profile $\rho(\cdot)$ such that $\alpha = \rho(0) \leq \rho(\cdot) \leq \rho(1) = \beta$, and $\rho(\cdot)$ is $\frac{\gamma}{2}$-Hölder at the boundaries, when $\gamma+\theta-1 = 0$, the Holder regularity at the boundary is $\frac{\gamma+1}{2}$. Since we imposed the same conditions as in the previous lemma in the profile $\rho(\cdot)$ then in this case (A.4.1) and (A.4.3) holds. From now on we suppose that $\gamma+\theta-1 > 0$, the other case is completely analogous. By the entropy and Jensen's inequalities, for any $B > 0$, the previous expectation is bounded from above by

$$\frac{H(\mu_N|\nu^N_{\rho(\cdot)})}{BN} + \frac{1}{BN}\log \mathbb{E}_{\nu^N_{\rho(\cdot)}}\Big[e^{BN\left|\int_0^t \overrightarrow{\eta}^{\varepsilon N}_{sN^2}(1)-\eta_{sN^2}(1)\,ds\right|}\Big].$$

By (A.1.2), the Feynman-Kac's formula, and using the same argument as in the proof of the previous lemma, the estimate of the previous expression can be reduced to bound

$$\frac{C_0}{B} + t\sup_f\Big\{\frac{1}{\ell}\sum_{y=2}^{\ell+1}|\langle \eta(y)-\eta(1), f\rangle_{\nu^N_{\rho(\cdot)}}| - \frac{N}{4B}\mathscr{D}_N(\sqrt{f},\nu^N_{\rho(\cdot)}) + \frac{C}{B}\sigma^2 + \frac{C\kappa\log(N)}{BN^{\gamma+\theta-1}}\Big\}, \tag{A.4.5}$$

where $\ell = \varepsilon N$. Here the supremum is carried over all the densities f with respect to $\nu^N_{\rho(\cdot)}$. Note that since $y \in \Lambda_N$ we know that

$$\eta(y)-\eta(1) = \sum_{z=1}^{y-1}(\eta(z+1)-\eta(z)).$$

Observe now that

$$\int(\eta(z+1)-\eta(z))f(\eta)d\,\nu^N_{\rho(\cdot)} = \frac{1}{2}\int(\eta(z+1)-\eta(z))(f(\eta)-f(\eta^{z,z+1}))d\,\nu^N_{\rho(\cdot)} + \frac{1}{2}\int(\eta(z+1)-\eta(z))(f(\eta)+f(\eta^{z,z+1}))d\,\nu^N_{\rho(\cdot)}.$$

By using the fact that for any $a,b \geq 0$, $(a-b) = (\sqrt{a}-\sqrt{b})(\sqrt{a}+\sqrt{b})$ and Young's inequality, we have, for any positive constant A, that

$$\begin{aligned}
&\frac{1}{\ell}\sum_{y=2}^{\ell+1}|\langle \eta(y)-\eta(1), f\rangle_{\nu^N_{\rho(\cdot)}}| \\
&\quad\leq \frac{1}{2A\ell}\sum_{y=2}^{\ell+1}\sum_{z=1}^{y-1}\int(\eta(z+1)-\eta(z))^2\left(\sqrt{f(\eta)}+\sqrt{f(\eta^{z,z+1})}\right)^2 d\,\nu^N_{\rho(\cdot)} \\
&\quad+\frac{A}{2\ell}\sum_{y=2}^{\ell+1}\sum_{z=1}^{y-1}\int\left(\sqrt{f(\eta)}-\sqrt{f(\eta^{z,z+1})}\right)^2 d\,\nu^N_{\rho(\cdot)} \\
&\quad+\frac{1}{2\ell}\sum_{y=2}^{\ell+1}\left|\sum_{z=1}^{y-1}\int\left(\eta(z+1)-\eta(z)\right)\left(f(\eta)+f(\eta^{z,z+1})\right)d\,\nu^N_{\rho(\cdot)}\right|.
\end{aligned} \tag{A.4.6}$$

Now, we neglect jumps of size bigger than one as we did below (A.3.4), from where we get that the second term on the right hand side of (A.4.6) is bounded from above by $CA\mathscr{D}_N(\sqrt{f}, \nu^N_{\rho(\cdot)})$ where C is a positive constant independent of A,ℓ,f. Then, for the choice $A = N(4BC)^{-1}$ and since $\gamma+\theta-1 \geq 0$, we can bound from above (A.4.5) by

$$\begin{aligned}
&\frac{2BC}{N\ell}\sum_{y=2}^{\ell+1}\sum_{z=1}^{y-1}\int(\eta(z+1)-\eta(z))^2\left(\sqrt{f(\eta)}+\sqrt{f(\eta^{z,z+1})}\right)^2 d\,\nu^N_{\rho(\cdot)} \\
&+\frac{1}{2\ell}\sum_{y=2}^{\ell+1}\left|\sum_{z=1}^{y-1}\int\left(\eta(z+1)-\eta(z)\right)\left(f(\eta)+f(\eta^{z,z+1})\right)d\,\nu^N_{\rho(\cdot)}\right|+\frac{C'}{B} \\
&\leq C\Big(\frac{B\ell}{N}+\frac{1}{B}+\frac{1}{2\ell}\sum_{y=2}^{\ell+1}\Big|\sum_{z=1}^{y-1}\int\left(\eta(z+1)-\eta(z)\right)\left(f(\eta)+f(\eta^{z,z+1})\right)d\,\nu^N_{\rho(\cdot)}\Big|\Big)
\end{aligned} \tag{A.4.7}$$

for some constant C. For the last inequality we used Lemma A.2.3. Observe that $B\ell/N = B\varepsilon$ vanishes as $\varepsilon \to 0$. It remains to estimate the third term on the right hand side of the last inequality. For that purpose we make a similar computation to the one of Lemma A.3.1 from where we get that

$$\sum_{z=1}^{y-1}\left|\int(\eta(z+1)-\eta(z))(f(\eta)+f(\eta^{z,z+1}))d\nu_{\rho(\cdot)}^N\right| \leq C\sum_{z=1}^{y-1}\left|\rho\left(\tfrac{z+1}{N}\right)-\rho\left(\tfrac{z}{N}\right)\right|.$$

Since $\rho(\cdot)$ is Lipschitz, by (A.4.7), this estimate provides an upper bound for (A.4.5) which is in the form of a constant times

$$\frac{B\ell}{N}+\frac{1}{B}+\frac{1}{N\ell}\sum_{y=2}^{\ell+1}y \ \leq B\varepsilon+B^{-1}+\varepsilon$$

which vanishes, as $\varepsilon\to 0$ and then $B\to\infty$. This ends the proof. $\square$

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