Asymptotic Behaviour of Exclusion Processes With Non-linear Boundary Dynamics

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Para a minha mãe preferida.

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Resumo

Nesta dissertação de mestrado consideramos o Processo de Exclusão Simples Simétrico numa caixa $\Lambda_N = \{1, \ldots, N-1\}$ acoplado com um reservatório de partículas não linear, lento, em cada extremidade, que injeta e retira partículas numa janela de tamanho K (*i.e.,* em $\{1, \ldots K\}$ e $\{N-K, \ldots, N-1\}$). Nessa janela, uma partícula entra no sistema no primeiro sítio livre, e sai apenas do primeiro sítio ocupado. Estes reservatórios induzem correlações entre as partículas, daí o termo não linear. As taxas de entrada e saída de partículas são proporcionais a $\kappa N^{-\theta}$, o que faz com que para $\theta > 0$ os reservatórios tenham uma ação lenta. Mostramos que a densidade espacial de partículas é descrita por uma solução fraca da equação do calor com condições de fronteira de Robin, se $\theta = 1$, ou de Neumann, se $\theta > 1$. A nossa dinâmica de fronteira é uma extensão do modelo com reservatórios de "corrente", e sua relevância está na sua generalização e tratamento das correlações.

Em seguida, estudamos a propagação do caos, através de cotas para as chamadas $v-$functions. Dizemos que existe propagação do caos quando temos que qualquer número finito de partículas evolui independentemente, quando o número total de partículas vai para infinito. Mostramos que o nosso modelo tem, de facto, essa propriedade.

Por fim, estudamos algébricamente o Matrix Product Ansatz para $K = 1$ no regime lento ($\theta \geq 0$), e extendemos a atual metodologia para $K = 2$. Para $K = 1$ e $\theta \neq 0$, fazemos uma pequena correção na algebra, e para $K = 2$ mostramos sob que condições a nossa algebra é consistente. Fomos bem sucedidos em induzir uma algebra consistente para taxas gerais em regime lento, exceto em um caso particular.

Palavras-chave: Limite Hidrodinâmico, Processo de exclusão, Equação do Calor, Dinâmica não linear, Propagação do Caos, Matrix Product Ansatz

Abstract

We consider the Symmetric Simple Exclusion Process in the box $\Lambda_N = \{1, \ldots, N-1\}$ coupled with non linear slow reservoirs at each endpoint, that injects and removes particles in a window of size K. A particle may enter to the first free site and leave from the first occupied site in its respective window (*i.e.,* $\{1, \ldots K\}, \{N - K, \ldots, N-1\}$). These reservoirs induce correlations between particles, hence the name non linear reservoirs. The rates of injection/removal are proportional to $\kappa N^{-\theta}$, thus for $\theta > 0$ the action of the reservoirs is slow. We show that the spatial density of particles is given by a weak solution of the heat equation with Robin boundary conditions, if $\theta = 1$, and Neumann boundary conditions, if $\theta > 1$. Our model is an extension of the "current reservoirs" model, and the main interest lies both in the generalization and the treatment of the correlation terms.

Next, we study the *propagation of chaos* through the estimation of $v-$functions. The propagation of chaos property states that any finite number of particles will evolve independently as the total number of particles goes to infinity. We will show that this indeed holds for our model.

At last, we study algebraically the Matrix Product Ansatz method for $K = 1$ under the slow/fast regime, and make a small extension of the current methodology for $K = 2$. For $K = 1$ and $\theta \neq 0$ we make a small correction in the current algebra, and for $K = 2$ we show under which conditions our algebra is consitent. When consistent, the normalization constant satisfies a second order recurrence. Our formulation was successfull in inducing a consistent algebra for general and slow rates, except for a particular case.

Keywords: Hydrodynamic Limit, Exclusion process, Heat equation, Non linear dynamics, Propagation of Chaos, Matrix Product Ansatz

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Chapter 1

Introduction

An *Interacting Particle System* is a mathematical model involving very-many components that interact with each other. Given the large number of particles and their possible interactions, the most natural mathematical frame to study these models will not be deterministic, but probabilistic. In the context of this master's thesis, an interacting particle system can be seen as what you get by watching a macroscopic, deterministic system *through the looking glass* – what was fixed, now has a random nature, and what didn't move, now jumps and interacts. In the miscroscopic level, we have a discrete system that evolves in time according to random clocks under some interaction among particles. The study of how these interactions affect the macroscopic level, that is, the passage from the micro to the macro, or the passage from the discrete to the *continuum*, is a central question in *Statistical Mechanics*. The rigorous study of *Interacting Particle Systems* is quite recent, having started with *Frank Spitzer*[1] in the seventies. The limiting object of this microscopic system is often described as the solution of a *Partial Differential Equation*. In this thesis, our macroscopic element is the *Heat Equation* with Robin boundary conditions, or Neumann boundary conditions.

The dynamics studied in this thesis is a generalization of the dynamics studied in [24]. We consider the Symmetric Simple Exclusion in the bulk, and both injections and removals of particles in a fixed window at the endpoints of the bulk– all these terms will be explained in the following sections. Moreover, we consider the "frequency" a particle is added/removed in the system dependend on a parameter θ. In this way, not only our model is a generalization of [24], but as a particular case we have also a regime not yet studied in the aforementioned work.

This thesis is divided in 6 chapters. We start with the mathematical background, in order to provide the reader the mathematical context and tools necessary for a better understanding of the following chapters. Then, in Chapter 3 we show the *Hydrodynamic Limit*– more specifically, in Section 3.3. That chapter is divided in 3 sections: first we define the dynamics through the *infinitesimal generator*, then we proceed with the *heuristics for the Hydrodynamic Limit*. This section is important because it allows us to understand the difficulties of the formal proof and have some insight to what are the induced *Hydrodynamic Equations* by our model. We then show the *Hydrodynamic Limit*, whose proof has 2 main steps: *tightness*, proved in Section 3.3.1, and *characterization of limit points*, proved in Section 3.3.2. For the characterization of limit points, we will need some results, which we

[1] Frank Spitzer is one of the pioneers in the rigorous study of an interacting particle system in the stochastic envoirnment. For more references of the context where the field of interacting particle systems emerged, see [31].

postpone the proofs to its own subsections: *Replacement Lemmas*, proved in Appendix A, and *Energy Estimate*, proved in Appendix B.

In chapter 4, we estimate the correlations, through the estimation of the, so called, v-functions. This chapter is both a detailed study of [24] and a simple adaptation of the arguments to show the bounds for the correlations in the *slow regime*. This chapter is divided in 5 sections: we start with the notation for the chapter and some definitions, where we define a coupling with a process of independent particles, in Section 4.2 we derive integral inequalities for the $v-$functions (that measure how "far" our system is from a system with independent particles), in the following Section, 4.3, we define the *truncated hierarchy* and the *branching process*, that classify the terms arising in the bounds for the $v-$functions in terms of a process denoted by *skeleton*, in Section 4.4 we derive bounds for the *skeleton*, and in last section, 4.5, we find estimates for the $v-$functions.

Finally, in last chapter we study the *Matrix Product Ansatz* (MPA). This chapter is divided in 3 sections. In the first, we present the *mathematical framework* of the MPA and do a review of the known results, difficulties and incapacities of the method. The second section is named *A new look in the linear SSEP*, where we study deeply the *linear SSEP* with general rates from an *algebraic point of view*. In this section we make a small correction to the known algebra in the *slow regime*. In the last section, we propose a generalization of the method for boundaries acting on 2 sites each. This section consistis in the statement of the generalization, and proof of the consistency of the *algebra*. In particular, we show that, in our framework, the *normalization constant* is *at best* a second order recurrence. Given the extent of this thesis, we will only do the essential computations and state the main results. In this section, we also introduce some new definitions in our *algebraic context*, which can easily be adapted for the general setting. To our knowledge, this is the first successful, *up to now*, generalization in this direction.

The exposition will be very compact and result oriented. We will only give a mathematical context in the first chapter, and the main theoretical results needed through the proofs will be stated in the *appendix*. In this way, we focus in a more compact and fluid exposition.

Chapter 2

Mathematical Background

2.0.1 Markov processes

A *probability space* is a well defined *measure space* with a normalized and positive measure - a *probability*. We write our probabilty space as a triple $(\Omega, \mathcal{F}, P)$ where

1. Ω is the *sample space* $-$ a non-empty set;
2. $\mathcal{F}$ the set of *events* $-$ a $\sigma-$algebra on Ω;
3. $P : \mathcal{F} \to [0,1]$ is a *probability measure* with the properties:

   $\sigma-$aditive : for $A_1, A_2, \dots$ disjoint sets, $P(\bigcup_{i\geq 1} A_i) = \sum_{i\geq 1} P(A_i)$

   normalized: $P(\Omega) = 1$.

Considering our probability space $(\Omega, \mathcal{F}, P)$ and $(S, \mathcal{S})$ a measurable space (where S is a non-empty set and $\mathcal{S}$ a $\sigma-$algebra), we define an $(S, \mathcal{S})-$valued *random variable* as a measurable function $X : \Omega \to S$. Thus, $\forall$ subset $B \in \mathcal{S}$ we have $X^{-1}(B) \in \mathcal{F}$, where $X^{-1}(B) = \{\omega \in \Omega : X(\omega) \in B\}$. In this way, the measure of a set (probability of an event) is defined as the measure of the pre-image by our function, the random variable: $P_X(B) = P(X^{-1}(B)) = P(\omega \in \Omega : X(\omega) \in B)$.

Taking time into consideration we define a *continuous-time stochastic process* $X = \{X_t\}_{t\geq 0}$ as a family of random variables indexed in the time t taking values in some space S with a metrizable structure given by the *Borel* $\sigma-$algebra, which is called the *state space* of the process. For fixed $T > 0$, we define the *path space* for our process on S by the set of *right continuous functions with left limits* (denoted by *càdlag* [1])

$$D_S[0,T] = \{X_\cdot : [0,T] \longrightarrow S, \text{càdlag}\} \tag{2.0.1}$$

also known as the set of *realizations* of our process. Thus, fixed a time t, $X_t : \Omega \to S$ is a *random-variable*. To define a measurable structure on $D_S[0,T]$, we will consider the smallest σ-algebra on $D_S[0,T]$, $\mathcal{F}$, such that the maps $X_\cdot \mapsto X_s$ are measurable with respect to $\mathcal{F}$, that is:

$$\{X_s \in A\} = \{X_\cdot \mid X_s \in A\} \in \mathcal{F}, \qquad \forall A \in S \text{ measurable.} \tag{2.0.2}$$

[1] From the french *"continue à droite, limite à gauche."*

Given that our process is a function of two variables $X_t(\omega)$, we want to construct the analogous of our $\sigma-$algebra with respect to the time variable. If we have that $\mathcal{F}$ is the smallest σ-algebra on our canonical path space such that $\forall s \leq t$, $X_. \mapsto X_s$ are measurable maps, then the collection $\{\mathcal{F}_t\}_{t\geq 0}$ is called the *natural filtration* of our process. Defined our *filtered space* $(D_S[0,T], \mathcal{F}, (\mathcal{F}_t)_{t\geq 0})$ we can now define our stochastic process on the *probability space* $(D_S[0,T], \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$.In our context, the stochastic process $X = \{X_t\}_{t\geq 0}$ is going to be a jump process. As in [17] we refer to P as the probability measure on S and $\mathbb{P}$ as the probability measure on $D([0,T],S)$, and analagously we differ E and $\mathbb{E}$ to denote the expectation with respect to P and $\mathbb{P}$, respectively. In particular, we have that $\mathbb{E}X_0 = EX$.

As defined in [7], a *Markov process* is a stochastic process whose future behavior depends only on the past through its present state; or the past depends on the future through the present; or even: given the present, the future and the past are independent. There are very different constructions of *Markov processes*: the classic construction from the Poisson process, as in [15] or [13], but one can also define it through the semigroup or generator (which we will present on the following sections). Here we present a brief "axiomatic" definition for the *time homogenous Markov process* from [13], and proceed to a more detailed overview on *generators* and *semigroups* and the construction of the *Markov semigroup* from the Markov *transition function*, as in [7].

A Markov process whose distribution at time t, given that at time s it was in ξ, depends only on the time lag $(t-s)$ and not on the specific times itself (condition 3. of the following definition) are termed *time homogenous Markov processes*:

Definition 2.0.1. A *time homogenous* Markov process on S is a collection $\{\mathbb{P}^\xi\}_{\xi\in S}$ of probability measures on $D[0,T]$ with the properties:

1. $\mathbb{P}^\xi(\eta_. \in D[0,T] : \eta_0 = \xi) = 1$ for all $\xi \in S$, that is, $\mathbb{P}^\xi$ is normalized, given the initial condition ξ;

2. The map $\xi \mapsto \mathbb{P}^\xi(A)$ is measurable $\forall A \in \mathcal{F}$;

3. $\mathbb{P}^\xi(\eta_{+.} \in A \mid \mathcal{F}_t) = \mathbb{P}^{\eta_t}(A)$ $\forall \xi \in S, A \in \mathcal{F}, t > 0$. (This is the *Markov property*).

We present next what is meant by the *transition function* of such processes.

Definition 2.0.2. The *transition function* of a time-homogeneous *Markov process* is a function $K(t,\xi,B)$ (K stands for *Kernel*), where $t \geq 0, \xi \in S, B \in \mathcal{F}$ with $(S,\mathcal{F})$ measurable space, S is the set of possible values of the process. K satisfies the properties:

1. $K(t,\xi,\cdot)$ is a probability measure on $(S,\mathcal{F})$, $\forall t \geq 0, \xi \in S$;

2. $K(0,\xi,\cdot) = \mathbb{1}_\xi$;

3. $K(t,\cdot,B)$ is measurable $\forall t \geq 0$ and $B \in \mathcal{F}$;

4. The *Chapman-Kolmogorov* equation (*CK-equation*) is satisfied:

$$\int_S K(s,\xi',B)K(t,\xi,d\xi') = K(t+s,\xi,B), \tag{2.0.3}$$

where $\mathbb{1}_{\cdot}$ is the indicator function. Moreover, we say that a stochastic process X on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with values in S is a time homogenous Markov process with transition function K if for $t > s$ we have:

$$P(X_t \in B \mid X_s) = K(t-s, X_s, B), \qquad B \in \mathcal{B}(S), \tag{2.0.4}$$

where $\mathcal{B}(S)$ is the *Borel $\sigma-$algebra* on S.

From the definition we can see that $K(t, \xi, \cdot)$ is the distribution of the position of the process at time t given that at the begining ($t = 0$) it was in ξ. To make the notion of transition function clearer, given a transition function K on a discrete space (we will take for example $S \subset \mathbb{N}$), we can define for $m, n \in S$ $p_{n,m}(t) := K(t, n, \{m\})$ for $t \geq 0$. This way, we have

- $\mathbb{1}_n(\{n\}) = K(0, n, \{n\}) = p_{n,n}(0) = 1$;
- $\mathbb{1}_n(\{m\}) = K(0, n, \{m\}) = p_{n,m}(0) = 0$ for $n \neq m$;
- $\forall t \geq 0 \quad p_{n,m}(t) \geq 0$ from the definition of probability measure;
- $1 = K(t, n, S) = \sum_{m \in S} K(t, n, \{m\}) = \sum_{m \in S} p_{n,m}(t) = 1$;

and the CK-equation

$$\sum_{k \in S} K(s, n, \{k\}) K(t, k, \{m\}) = K(t+s, n, \{m\}) \tag{2.0.5}$$

takes now the form $p_{n,m}(s+t) = \sum_{k \in S} p_{n,k}(s) p_{k,m}(t)$. On this simple case, one may easily do even better and show directly that, by conditioning, the CK-equation is satisfied:

$$\begin{aligned} P(X_{s+t} = m \mid X_0 = n) &= \sum_{k \in S} P(X_{s+t} = m \mid X_t = k, X_0 = n) P(X_t = k \mid X_0 = n) \\ &= \sum_{k \in S} P(X_{s+t} = m \mid X_t = k) P(X_t = k \mid X_0 = n) \\ &= \sum_{k \in S} P(X_s = m \mid X_0 = k) P(X_t = k \mid X_0 = n), \end{aligned} \tag{2.0.6}$$

where for the last two equalities we used the Markov property, and the time-homogenous property, respectively. On the discrete case, these transition probabilities can naturally be expressed in terms of (possibly infinite) matrices defined with the usual matrix notation $P(t) = (p_{n,m}(t))_{n,m \geq 1}$. In this way we can also write (2.0.5) as $P(s)P(t) = P(s+t)$.

These matrices are termed *transition matrices*, and the above property is the *semigroup property*, which we will explore, in no time, for more general state spaces. Naturally, one also gets that $P(0)P(t) = P(0+t) \implies P(0) = 1$, where 1 is the identity matrix, which is coherent with our definition of transition function. For a clearer understanding of these matrices let us first consider the simplest case. Suppose the transitions do not depend on time, and both our state space and time are discrete. In this case, we have

$$P(n)P(m) = P(n+m) \Leftrightarrow P^n P^m = P^{n+m} \tag{2.0.7}$$

where P^n (resp. P^m) is the transition matrix to the power of n (resp. m), which comes directly from the time independence and the CK-equation. Looking directly to the transitions, conditioning on the past states we have

$$P(X_n = j) = \sum_{k \in S} P(X_0 = k) P(X_n = j \mid X_0 = k). \tag{2.0.8}$$

Note that $P(X_n = j \mid X_0 = k) = P^n_{k,j}$, and writing $P(X_0 = k) \equiv \alpha_k$ we have, for $j \in S$, $\sum_{k \in S} \alpha_k P^n_{k,j}$, and in matrix notation: $P(X_n = j) = (\alpha P^n)_j$.

This simple case is very important because it gives some intuition to what comes in the next sections. We showed that, for the full-discrete case, the distribution at a time n equals to the matrix product of the initial distribution and the transition matrix to the power of the time. That is, the transitions take us from the initial state to where we want to go; or in other words, where we are is a result from where we have started and the path we took. In fact, this holds for general topologies and continuous time as well. As stated in [27]:

Proposition 2.0.3. *A discrete time Markov chain $\{X_n\}_{n \in \mathbb{N}}$ is fully characterized by its transitions matrices and the probability function of X_0.*

The jump from discrete time to continuous time is not trivial, and we refer the reader to a very detailed construction on [15] using an embedded discrete time chain and the *Poisson Process*.

2.0.2 Generators and Semigroups

Now we shall give a brief introduction to semigroups and its generators. We will start with the definition of a semigroup, proceed to prove some of its general properties and define its generator at the end. Next, we will introduce the *Markov semigroup*, and as a motivation for the two main formulas used in this thesis, we will refer to the *Cauchy problem*. As a by-product of the Cauchy problem (but more on the perspective of the *Martingale problem*) we will construct a great tool for this dissertation: the Dynkin's formula.

Definition 2.0.4. Let $\mathcal{X}$ be a Banach space, and let $S_t, t \geq 0$ be bounded operators acting on $\mathcal{X}$, indexed in t. The family $\{S_t\}_{t \geq 0}$ is termed a strongly continuous (or of class c_0) *semigroup* of operators iff

1. $\forall s, t \geq 0, S_{s+t} = S_t S_s$;
2. $S_0 = 1_{\mathcal{X}}$, where $1_{\mathcal{X}}$ is the identity operator in $\mathcal{X}$;
3. $\lim_{t \to 0} S_t f = f$ for $f \in \mathcal{X}$.

If only the first two properties are satisfied, the family of operators is termed *semigroup*. On the above definition, we mean convergence on the usual *sup norm*.

An important property which will allow us to differentiate in time and we will not prove [7] is:

Proposition 2.0.5. *[Continuity]*
If $\{S_t\}_{t \geq 0}$ is of class c_0, then $\forall f \in \mathcal{X}$ the function $t \to S_t f$ is strongly continuous in $\mathbb{R}^+$ (right continuous at 0).

Closely related to the semigroup associated to a process is the *infinitesimal generator*, which can be interpreted as its derivative at time 0.

Definition 2.0.6. [Infinitesimal Generator] Let $\{S_t\}_{t\geq 0}$ be of class c_0 acting on a Banach space $\mathcal{X}$. Then if the following limit exists

$$\mathcal{L}f := \lim_{h\to 0^+} \frac{S_h f - f}{h} \tag{2.0.9}$$

$\mathcal{L}$ is termed the *infinitesimal generator* of the semigroup $\{S_t\}_{t\geq 0}$, whose domain is:

$$\mathcal{D}(\mathcal{L}) = \{f \in \mathcal{X} : \exists \lim_{h\to 0^+} \frac{S_h f - f}{h} =: \mathcal{L}f\} \tag{2.0.10}$$

The infinitesimal operator $\mathcal{L}$ is a linear operator [7], but this will be clear once we move to Markov semigroups). An important technical property proved in [7] is the following.

Proposition 2.0.7. *$\mathcal{D}(\mathcal{L})$ is a dense subset of $\mathcal{X}$.*

Naturally, one can integrate back (2.0.9) to get the following proposition.

Proposition 2.0.8. *Let $f \in \mathcal{X}$. For fixed $t \geq 0$ we have $\mathcal{L}\int_0^t S_s f ds = S_t f - f$.*

Proof. We want to show that

$$\mathcal{L}\int_0^t S_s f ds = \lim_{h\to 0^+} \frac{S_h \int_0^t S_s f ds - \int_0^t S_s f ds}{h} = S_t f - f \tag{2.0.11}$$

and that $\int_0^t S_s f ds \in \mathcal{D}(\mathcal{L})$. Note that

$$S_h \int_0^t S_s f ds = \int_0^t S_{s+h} f ds = \int_h^{t+h} S_s f ds = \int_0^{t+h} S_s f ds - \int_0^h S_s f ds, \tag{2.0.12}$$

where we used the semigroup property in the first equality, a change of variables in the second, and the last follows by the definition of the integral. Thus,

$$\begin{aligned} h^{-1}\left(S_h \int_0^t S_s f ds - \int_0^t S_s f ds\right) &= h^{-1}\left(\int_0^{t+h} S_s f ds - \int_0^h S_s f ds - \int_0^t S_s f ds\right) \\ &= h^{-1}\int_t^{t+h} S_s f ds - h^{-1}\int_0^h S_s f ds. \end{aligned} \tag{2.0.13}$$

Using the strong continuity of $t \to S_t f$ and the fundamental theorem of calculus we have, as $h \to 0^+$

$$\frac{1}{h}\int_t^{t+h} S_s f ds - \frac{1}{h}\int_0^h S_s f ds \to S_t f - S_0 f = S_t f - f. \tag{2.0.14}$$

Since everything is bounded, we conclude that $\int_0^t S_s f ds \in \mathcal{D}(\mathcal{L})$. ☐

Extending the idea of the *generator* as a *differential operator*, but now up to a time t, we have the following proposition:

Proposition 2.0.9. *If $f \in \mathcal{D}(\mathcal{L})$, then $S_t f \in \mathcal{D}(\mathcal{L})$. Moreover, the function $t \to S_t f$ is continuously differentiable* *in* $\mathbb{R}^+$*, whith right derivative at* $t = 0$*, and*

$$\tfrac{dS_t f}{dt} = \mathcal{L}S_t f = S_t \mathcal{L} f, \qquad t \geq 0. \tag{2.0.15}$$

Proof.

$$\begin{aligned} & f \in \mathcal{D}(\mathcal{L}) \Leftrightarrow \mathcal{L}f := \lim_{h \to 0^+} \frac{S_h f - f}{h} < \infty \\ & \Rightarrow \mathcal{L}S_t f = \lim_{h \to 0^+} \frac{S_h S_t f - S_t f}{h} = \lim_{h \to 0^+} \frac{S_{t+h} f - S_t f}{h} \\ & = \lim_{h \to 0^+} S_t \frac{S_h f - f}{h} = S_t \lim_{h \to 0^+} \frac{S_h f - f}{h} = S_t \mathcal{L} f < \infty \\ & \Longrightarrow S_t f \in \mathcal{D}(\mathcal{L}). \end{aligned} \tag{2.0.16}$$

Moreover, since $\frac{dS_t f}{dt} := \lim_{h \to 0} \frac{S_{t+h} f - S_t f}{h}$ we have our result. □

We remark that $\frac{dS_t f}{dt} = \mathcal{L}S_t f$ is termed the *forward equation*, and $\frac{dS_t f}{dt} = S_t \mathcal{L} f$ the *backward equation*. The names come from the forward and backard Kolmogorov equations, which we will end this chapter with. Note also that the above proposition suggests that the *semigroup* is some sort of exponential function: $S_t = e^{t\mathcal{L}} = 1 + t\mathcal{L} + o(t)$.This turns out to be true in a certain sense, which is made formal on the Hille–Yosida–Feller–Phillips–Miyadera theorem, also known as *Hille-Yosida* theorem.[2] Before proceeding to the statement of *Hille-Yosida's* theorem, we define the semigroup and infinitesimal generator in the context of *Markov processes*.

Theorem 2.0.10 (Hille-Yosida)**.** *There exists a one-to-one correspondence between Markov generators and semigroups on* $C(S) = \{f \text{ continuous } \mid f : S \to \mathbb{R}\}$*, given by (2.0.9) and:*

$$S_t f = e^{t\mathcal{L} f} := \lim_{n \to \infty} (1 - \frac{t}{n}\mathcal{L})^{-n} f, \qquad \textit{for } f \in C(S), t \geq 0. \tag{2.0.17}$$

As explored in [7], this result is consequence of the *Yosida approximation* of operators, and usually expressed in terms of its resolvent. The proof presented there requires the *Laplace transformation* of the semigroup, which is not in the context of this work. Note that if we consider $S_t f \equiv f_t$, and remembering the interpretation of the generator as a *differential operator*, Proposition 2.0.9 states that $S_t f$ is the solution to some sort of differential equation. In fact, that equation is known as the *Cauchy problem*, and it is stated as follows:

Proposition 2.0.11 (Semigroups and the Cauchy problem)**.** *Let* $\mathcal{L}$ *be the generator of a strongly continuous semigroup* $\{S_t\}_{t \geq 0}$ *acting in a Banach space* $\mathcal{X}$*. The Cauchy problem*

$$\begin{cases} \frac{df_t}{dt} = \mathcal{L} f_t, \qquad t \geq 0 \\ f_0 = f \in \mathcal{D}(\mathcal{L}) \end{cases} \tag{2.0.18}$$

where f_t *is a sought-for differentiable function with values in* $\mathcal{D}(\mathcal{L})$*, has the unique solution* $f_t = S_t f$*.*

[2]The original formulation was given independently by Hille and Yosida, and the general case was discovered later, independently, by Feller, Phillips and Miyadera.

To prove Proposition 2.0.11, we only need to show uniqueness, since the solution follows from Proposition 2.0.9.

Note also that Proposition 2.0.11 is a very important and general result. Considering, for example, the standard *Brownian motion*, we have that all the Brownian motion properties are "hidden" on the operator ∂_u^2, and the Cauchy problem takes the form of the well known heat equation. Now that we have done an introduction to semigroups and generators, let us go back to *Markov processes* and define more explicitly what we mean by *semigroup* in this context.

With a transition function, one may associate a family of operators in $\mathbb{BM}(S)$- the set of bounded Borel charges (signed measures) on S (for our purposes, one can consider $\mathcal{P}(S)$ - the set of probability measures endowed on S with weak convergence) by

$$(U_t\mu)(B) = \int_S \mu(dp)K(t,p,B). \tag{2.0.19}$$

From the definition above, one can check that $U_t\mu$ is, in fact, a measure. In particular, if μ is a *probability measure*, then $U_t\mu$ is also a probability measure. Moreover, by the CK-equation, $\{U_t\}_{t\geq 0}$ is a semigroup of operators:

$$\begin{aligned}(U_t\mu)(B) &= \int_S \mu(dp)\int_S K(l,q,B)K(s,p,dq)\\ &= \int_S\int_S \mu(dp)K(s,p,dq)K(l,q,B)\\ &= \int_S (U_s\mu)(dq)K(l,q,B)\\ &= (U_sU_l\mu)(B)\end{aligned} \tag{2.0.20}$$

where $t = s + l$, and $U_0\mu = \int_S \mu(dp)K(0,p,B) = 1$.The main reason for introducing this formula is for the following interpretation: if X_t is a *Markov process* with transition K and initial distribution μ, then $U_t\mu$ is the distribution of the process at time t. That is, given a measure μ, the semigroup operator is what "makes the time running": $U_t\mu \equiv \mu_t$. However, the semigroup we are going to consider is the dual of (2.0.19):

$$S_tf(p) = \int_S f(q)K(t,p,dq) \qquad t \geq 0 \tag{2.0.21}$$

defined in $BM(S)$-space of bounded measurable functions on S . When we say dual, we mean that we may treat a member of $BM(S)$ as a functional on $\mathbb{BM}(S)$ (the dual space), given by $\mu \mapsto \int_S f d\mu =: \langle f, \mu\rangle$, and $\langle f, U_t\rangle = \langle S_tf, \mu\rangle$. That is, the dual of U_t equals S_t on $BM(S)$. An important remark is related to a class of processes termed *Feller processes*, which we shall introduce on the following definition:

Definition 2.0.12. A Markov process (semigroup) is a Feller process (semigroup) if $f \in C_b(S)$ (continuous and bounded functions) $\Rightarrow S_tf \in C_b(S) \quad \forall t \geq 0$. That is, S_t leaves $C_b(S)$ invariant.

Given that $\langle f, U_t\rangle = \langle S_tf, \mu\rangle$, $U_t\mu \equiv \mu_t$, and the *Feller semigroup* is a bounded operator that maps $C_b(S)$ into itself, one may ask what is the action of the *adjoint* of S_t. That is, $\langle f, U_t\rangle = \langle S_tf, \mu\rangle = \langle f, S_t^*\mu\rangle$, and what is its relation to the distribution of the process. There are two answers to this question: one relating to the original process, and the other related to the time-reversed process . The second answer we will explain in the following section relating *invariant measures*, and the first is given on [20] as a definition:

Definition 2.0.13. For a process $\{S(t)\}_{t\geq 0}$ with initial distribution μ we denote $\mu S(t) \in \mathcal{P}(X)$ the distribution at time t, which is uniquely determined by

$$\int_S f d[\mu S_t] := \int_S S_t f d\mu, \qquad \text{for all } f \in C(S). \tag{2.0.22}$$

Writting as follows makes it a little bit clearer:

$$\langle f, \mu S(t)\rangle = \int_S f d[\mu S(t)] := \int_S S(t) f d\mu = \langle S(t) f, \mu\rangle = \langle f, S(t)^* \mu\rangle. \tag{2.0.23}$$

Thus, when we see μS_t we are refering to the distribution at time t, yet it is implicit the adjoint of S_t: $S_t^* \mu \Leftrightarrow \mu S_t$. Given a Feller semigroup, where the dual and adjoint coincide, we can see that if the distribution is independent of time, then $S_t^* = S_t = S_0 = 1$. Finally, we will use the following expression for the *Markov semigroup*, given its more probabilistic notation:

$$(S_t f)(X_0) = \mathbb{E}[f(X_t) \mid X_0] \tag{2.0.24}$$

From the definition, we can interpret the *Markov semigroup* as the mean path of our process $\{f(X_t)\}_{t\geq 0}$ starting from X_0. We will show one last property of the generator related to Markov processes, since it will be useful on this dissertation for certain convergence results:

Proposition 2.0.14 (Time scalling of $\mathcal{L}$)**.** *Given a Markov process $\{X_t\}_{t\geq 0}$ with semigroup $\{S_t\}_{t\geq 0}$ and generator $\mathcal{L}$, if we change the time scale by a factor $\theta(n)$ then the generator of the process $\{\eta_{\theta(n)t}\}_{t\geq 0}$ is given by $\theta(n)\mathcal{L}$.*

Proof. $\theta(n)\mathcal{L}f := \lim_{h\to 0^+} \theta(n)\frac{S_h f - f}{h} = \lim_{t\to 0^+} \theta(n)\frac{S_{\theta(n)t} f - f}{\theta(n)t} = \lim_{t\to 0^+} \frac{S_{\theta(n)t} f - f}{\theta(n)t}$. □

2.0.3 Generators on a finite state space

Up to this point, we already gave two interpretations of the semigroup (in general, and for Markov processes), but the interpretation of the generator as a differential operator is still too abstract. Thus, let us take a look more closely to its action. We will denote processes on a finite state space, which is the context of this thesis, by lower case greek letters, for example, η_t instead of X_t. Note that our transition probability for $\eta = \xi \rightsquigarrow \eta_t = \xi'$ can be written as:

$$\mathbb{P}^\eta(\eta_t = \xi') = \mathbb{E}^\eta 1_{\eta_t}(\xi') = S_t 1_\eta(\xi'). \tag{2.0.25}$$

Then, by (2.0.9), for small t we have $\mathbb{P}^\eta(\eta_t = \xi') = 1_\eta(\xi') + t\mathcal{L}1_\eta(\xi') + o(t)$. Thus, $\mathcal{L}$ can be interpreted as a *transition rate*; probability per time unit. As in [13], the *rates* $c(\xi, \xi')$ are defined by

$$\mathbb{P}^\xi(\eta_t = \xi') = 1_\xi(\xi') + c(\xi, \xi')t + o(t) \quad \text{as } t \searrow 0. \tag{2.0.26}$$

For a finite state space, which is the case we are interested in, we have the following proposition.

Proposition 2.0.15. *If $\{\eta_t\}_{t\geq 0}$ is a Markov process on a countable state space S with jump rates $c(\eta,\eta')$ then for f cilindrical function (functions that are dependent on η through a finite number of coordinates), the infinitesimal generator has the following expression:*

$$\mathcal{L}f(\eta) = \sum_{\eta'\in S} c(\eta,\eta')(f(\eta') - f(\eta)). \tag{2.0.27}$$

Proof.

$$\begin{aligned} S_t f(\eta) &= \mathbb{E}^\eta f(\eta_t) = \sum_{\eta'\in S} f(\eta')\mathbb{P}^\eta(\eta_t = \eta') \\ &= \mathbb{P}^\eta(\eta_t = \eta)f(\eta) + \sum_{\eta'\neq\eta} f(\eta')\mathbb{P}^\eta(\eta_t = \eta') \\ &= (1 - \mathbb{P}^\eta(\eta_t \neq \eta))f(\eta) + \sum_{\eta'\neq\eta} f(\eta')c(\eta,\eta')t + o(t) \\ &= (1 - \sum_{\eta\neq\eta'} \mathbb{P}^\eta(\eta_t = \eta'))f(\eta) + \sum_{\eta'\neq\eta} f(\eta')c(\eta,\eta')t + o(t) \\ &= (1 - \sum_{\eta\neq\eta'} c(\eta,\eta')t)f(\eta) + \sum_{\eta'\neq\eta} f(\eta')c(\eta,\eta')t + o(t). \end{aligned} \tag{2.0.28}$$

That is, $S_t f(\eta) - f(\eta) = \sum_{\eta\neq\eta'} c(\eta,\eta')(f(\eta') - f(\eta))t + o(t)$, and by (2.0.9) the result holds. □

We can also define the *rate's matrix* by noticing that

$$\begin{aligned} 1 &= \sum_{\xi'\in S} \mathbb{P}^\xi(\eta_t = \xi') = 1 + t\sum_{\xi'\in S} c(\xi,\xi') + o(t) = 1 + t(c(\xi,\xi) + \sum_{\xi'\in S} c(\xi,\xi')) + o(t) \\ \Leftrightarrow 0 &= t(c(\xi,\xi) + \sum_{\xi'\in S} c(\xi,\xi')) + o(t), \end{aligned} \tag{2.0.29}$$

and we must have $c(\xi,\xi) + \sum_{\xi'\in S} c(\xi,\xi') = 0$. Identifying the diagonal as $c(\xi,\xi) = -\sum_{\xi'\neq\xi} c(\xi,\xi')$ one can easily construct the matrix given the rates of the process.

We can relate the above expression with (2.0.16), resulting on the *Master equation*, also known as *Kolmogorov's equation*. Taking $f = 1_\eta$ on (2.0.27):

$$\begin{aligned} \mu(S_t\mathcal{L}1_\eta) &= \int_S \sum_{\xi'\in S} S_t c(\xi,\xi')(1_\eta(\xi') - 1_\eta(\xi))d\mu(\xi) = \int_S c(\xi,\eta) - \sum_{\xi'\in S} c(\xi,\xi')1_\eta(\xi)d\mu S_t(\xi) \\ &= \sum_{\xi\in S}[\mu S_t](\xi)c(\xi,\eta) - [\mu S_t](\xi)1_\eta(\xi)\sum_{\xi'\in S} c(\xi,\xi') \equiv \sum_{\xi\in S} p_t(\xi)c(\xi,\eta) - p_t(\eta)\sum_{\xi'\in S} c(\eta,\xi') \\ &= \sum_{\eta'\in S}(p_t(\eta')c(\eta',\eta) - p_t(\eta)c(\eta,\eta')), \end{aligned} \tag{2.0.30}$$

where we defined $p_t := [\mu S_t]$ (recall (2.0.19)), and for the left hand-side:

$$\mu(\frac{d}{dt}S_t 1_\eta) = \frac{d}{dt}\mu(S_t 1_\eta) = \frac{d}{dt}[\mu S_t](1_\eta) \equiv \frac{d}{dt}p_t(\eta). \tag{2.0.31}$$

Thus,

$$\frac{d}{dt}p_t(\eta) = \sum_{\eta' \in S} (c(\eta', \eta)p_t(\eta') - p_t(\eta)c(\eta, \eta')). \tag{2.0.32}$$

We finish this section presenting two formulas that will be very useful in this dissertation. First, we present a martingale, termed *Dynkin's formula*, or *Dynkin's martingale*. This martingale will be useful in what follows because it will be the main tool to identity our process with a weak solution to the heat equation with specific boundary conditions.

Theorem 2.0.16 (Dynkin's formula)**.** *Let $\{\eta_t\}_{t\geq 0}$ be a Markov process with generator $\mathcal{L}$ and countable state space S, and $f : \mathbb{R}^+ \times S \longrightarrow \mathbb{R}$ bounded such that:*

1. $\forall \eta \in S, f(\cdot, x) \in C^2(\mathbb{R}^+)$*;*

2. $\exists C < \infty : sup_{s,\eta} \mid \partial_s^j f_s(\eta) \mid \leq C$ *for* $j = 1, 2$*;*

and for all $t \geq 0$ *let*

$$M_t^f = f_t(\eta_t) - f_0(\eta_0) - \int_0^t (\partial_s + \mathcal{L}) f_s(\eta_s) ds. \tag{2.0.33}$$

Then $\{M_t^f\}_{t\geq 0}$ is a martingale with respect to the natural filtration of $\{\eta_t\}_{t\geq 0}$.

A detailed proof can be found both in [17] and [15]. Since the expectation of a martingale is independent of time, we have that $\mathbb{E}M_t^f = \mathbb{E}M_0 = 0$. Moreover, we know that the process $N_t(F) := M_t(F)^2 - [M(F)]_t$ is a mean zero local martingale with respect to the natural filtration of $\{X_t\}_{t\geq 0}$. For *Dynkin's martingale* we have

$$N_t(F) := (M_t(F))^2 - \int_0^t B_s(F) ds \tag{2.0.34}$$

with $B_s(F) = \mathcal{L}F(s, X_s)^2 - 2F(s, X_s)\mathcal{L}F(s, X_s)$. For a more detailed analysis see [17].

Recalling (2.0.11), one can see that, intuitively, *Dynkin's formula* (2.0.16) looks like an integral expression of that differential equation (in the sense that $\mathcal{L}$ is a differential operator), extended to more general f functions. Also relating the *Cauchy problem* is the *Feynman-Kac formula*, which will be useful in the following, both in proving the *Replacement Lemmas* A.0.5 and A.0.6 and the energy estimate (Proposition B.0.4.) Consider a bounded function $V : \mathbb{R}^+ \times S \to \mathbb{R}$ satisfying the conditions of the Theorem 2.0.16, and a bounded function $F_0 : S \to \mathbb{R}$. For fixed $T > 0$ denote by $F : [0, T] \times E \to \mathbb{R}$ the solution of the differential equation

$$\begin{cases} (\partial_t u)(t, x) = (\mathcal{L}u)(t, x) + V(T - t, x)u(t, x), \\ u(0, x) = F_0(x). \end{cases} \tag{2.0.35}$$

Proposition 2.0.17. *The solution F of (2.0.35) has the following stochastic representation:*

$$F(T, x) = E_x \left[\epsilon^{\int_0^T V(s, X_s) ds} F_0(X_T). \right] \tag{2.0.36}$$

Again, this is a more general expression than the classical *Feynman-Kac* formula with respect to the *Brownian Motion*, where $\mathcal{L} = \triangle$, with $\triangle$ being the *laplacian*. The proof can be found in [17], page 342.

2.0.4 Stationary Measures

We end our mathematical presentation with a brief introduction to stationary measures. Recalling our discussion above, a measure $\mu \in \mathcal{P}(X)$ is *stationary/invariant* if $\mu S_t = \mu$, that is:

Definition 2.0.18 (Invariant measure)**.** The measure μ is invariant for the process "induced" by S_t iff $\int_S S_t f d\mu = \int_S f d\mu$ for all $f \in C(S)$.

On this section we will interplay between the following notations:

$$\mu(f) = \int f d\mu = \langle f, \mu \rangle. \tag{2.0.37}$$

The set of all invariant measures of a process is denoted by $\mathcal{I}$. Now we give the definition a *reversible* measure.

Definition 2.0.19. A measure μ is *reversible* iff $\mu(fS_tg) = \mu(gS_tf)$for all $f, g \in C(S)$.

Note that, taking $g = 1$, we can see that every reversible measure is stationary:

$$\langle fS_tg, \mu \rangle = \langle gS_tf, \mu \rangle \Leftrightarrow \langle fS_t1, \mu \rangle = \langle S_tf, \mu \rangle \Leftrightarrow \langle f, \mu \rangle = \langle S_tf, \mu \rangle. \tag{2.0.38}$$

Above we used that $S_t 1_\eta = 1$, which is true noting that $\mathbb{E}^\eta(1_\eta) = \mathbb{P}(\eta = \eta) = 1$.

Proposition 2.0.20. *Consider a Feller process on a compact state space S with generator $\mathcal{L}$. Then*

$$\mu \in \mathcal{I} \Leftrightarrow \mu(\mathcal{L}f) = 0 \qquad \forall f \in C_0(S), \tag{2.0.39}$$

where $C_0(S)$ is the set of cylindrical functions $f : S \to \mathbb{R}$.

Proof. $(\longrightarrow)$:

$$\mu(\mathcal{L}f) = \langle \mathcal{L}f, \mu \rangle = \langle S_t\mathcal{L}f, \mu \rangle = \langle \frac{dS_tf}{dt}, \mu \rangle = \frac{d}{dt}\langle S_tf, \mu \rangle = \frac{d}{dt}\int f d\mu = 0. \tag{2.0.40}$$

Note that we can exchange the integral and the derivative by the Leibniz rule, since S_tf is bounded. For the converse,

$(\longleftarrow)$:

$$\begin{aligned} 0 = \langle \mathcal{L}f, \mu \rangle = \langle S_t\mathcal{L}f, \mu \rangle = \int_0^t \langle S_s\mathcal{L}f, \mu \rangle ds = \int_0^t \int S_s\mathcal{L}f ds d\mu \\ = \int (\mathcal{L}\int_0^t S_s f ds) d\mu = \int (S_tf - f) d\mu = \int S_t f d\mu - \int f\mu. \end{aligned} \tag{2.0.41}$$

Above we used Fubini's theorem and Proposition 2.0.9. To show that the result holds $\forall f \in C_0(S)$, remember that by Propposition 2.0.7 $\mathcal{D}(\mathcal{L})$ is dense in $C_0(S)$ thus we can prove by the weak convergence of sequences of functions. ☐

Relating the *backward* and *forward* equations (2.0.9) with the stationary measure we can get the following condition:

$$\begin{aligned}\frac{d}{dt}S_t f = S_t\mathcal{L}f \Rightarrow \mu(\frac{d}{dt}S_t f) = \mu(S_t\mathcal{L}f)\\ \Leftrightarrow \frac{d}{dt}\mu(S_t f) = \mu(S_t\mathcal{L}f) \Leftrightarrow \frac{d}{dt}\mu(f) = \mu(\mathcal{L}f) = 0.\end{aligned} \tag{2.0.42}$$

The above relation will be very important on our study of the stationary measures through the *Matrix Product Ansatz*, described in the Chapter 5.

Now we make some remarks about the adjoint operator. Let $\mu \in \mathcal{P}(S)$ be the stationary measure of the process "induced" by $\{S_t\}_{t\geq 0}$, and let $L^2(S,\mu) = \{f \in C(S) : \mu(f^2) < \infty\}$. It can be shown that S_t and $\mathcal{L}$ have an adjoint operator [17], uniquely defined by:

$$\langle S_t g, f\rangle = \langle g, S_t^* f\rangle \quad \text{and} \quad \langle \mathcal{L}g, f\rangle = \langle g, \mathcal{L}^* f\rangle \qquad \forall f, g, \in L^2(S,\mu). \tag{2.0.43}$$

To compute the action of the adjoint operator, $\forall g \in L^2(S,\mu)$:

$$\langle g, S_t^* f\rangle = \int_S f S_t g d\mu = \mathbb{E}^\mu(f(\eta_0)g(\eta_t)) = \mathbb{E}^\mu(\mathbb{E}(f(\eta_0 \mid \eta_t))g(\eta_t)), \tag{2.0.44}$$

where in the last equality $\mathbb{E}^\mu$ denotes the expectation with respect to $\mathbb{P}\mu$. Since μ is stationary,

$$\begin{aligned}\mathbb{E}^\mu(\mathbb{E}(f(\eta_0 \mid \eta_t))g(\eta_t)) = \int_X \mathbb{E}(f(\eta) \mid \eta_t = \xi)g(\xi)\mu(\xi) = \mu(g\mathbb{E}(f(\eta_0) \mid \eta_t = \cdot))\\ = \langle g, E(f(\eta_0) \mid \eta_t = \cdot)\rangle,\end{aligned} \tag{2.0.45}$$

that is, $S_t^* f = E(f(\eta_0) \mid \eta_t = \cdot)$. In other words, the adjoint operator of the semigroup is the semigroup of the reversed process. One can also show that $\mathcal{L}^*$ is the generator of the adjoint semigroup [13] and the process is time-reversible if the generator and semigroup operators are *self-adjoint*. This is actually an intuitive result: if moving forward or backwards in time gives the same result, then moving in time *probably* does not matter.

Chapter 3

Hydrodynamic Limit

In this Chapter we show the Hydrodynamic Limit. The Hydrodynamic Limit states that the spacial density of the particle system studied in the following sections can be described, in the continuum setting, by a weak solution of the heat equation. In the first section we will introduce our model in terms of *Poisson clocks*. Then, we will define its generator, with the tools presented in the mathematical background. In the following section, we will state the main ideia of the Hydrodynamic Limit and do an heuristic to have some insight of the partial differential equation the mentioned spacial density is solution of. The main tool will be Dynkin's Martingale, already presented in Theorem 2.0.16. Then we will formally show the Hydrodynamic Limit. Through the proof, we will need two main results: Replacement Lemmas A.0.5 and A.0.6, and the energy estimate, in the Appendix B, to which we will direct the reader to the appendix for their proof.

For the main proofs of this chapter, we take results from [11], [10], [4] and [3]. These references are a good start for the interested reader. Moreover, for a deeper review, regarding other topics of interacting particle systems, such as fluctuations and propagation of chaos, and many other techniques, we refer the reader to [17], [23] and [22].

3.1 Dynamics

3.1.1 The model

The Symmetric Simple Exclusion Process (SSEP) in contact with *slow non-linear* reservoirs can be described as follows. Fixed a scalling parameter N, we denote our process by η_t and η is named the *configuration*. Our process lives in the discrete space $\{0,1\}^{\Lambda_N}$, where $\Lambda_N = \{1,\dots,N-1\}$. Each element $x \in \Lambda_N$ is called *site*, and Λ_N the *bulk*. The map $x \mapsto \eta(x)$ gives the number of particles on the site x. In our model, $\eta(x) \in \{0,1\}$. Each pair $\{x, x+1\}$ with $x \in \{1,\dots,N-2\}$ is named a *bond*. We artificially add two sites at end points of the *bulk* (the sites 0 and N) $-$ this gives two *bonds* ($\{0,1\}$ and $\{N-1,N\}$) $-$ these artificial sites are named *reservoirs*, and will contain an infinite number of particles. To each bond in the *bulk*, we associate a *Poisson process* $N_{x,x+1}(t)$ with parameter $\eta(x)(1-\eta(x+1)) + \eta(x+1)(1-\eta(x))$. For all $x \in \{1,\dots,N-2\}$ the associated *Poisson processes* are independent. These Poisson processes are named *Poisson clocks*, given that they dictate when a particle may jump, as we will explain shortly. Note that the probability of two clocks ringing at the same time is 0

(with respect to the Lesbesgue measure), since the exponential random variables are continuous and independent. From the rate of the clocks, one can see that the bulk follows an *exclusion rule*, hence the name *exclusion process*: if a clock at $\{x, x+1\}$ rings at time t and $\eta_t(x) = 0, \eta_t(x+1) = 1$ (resp. $\eta_t(x) = 1, \eta_t(x+1) = 0$) the particles exchange sites: $\eta_{t^+}(x) = 1, \eta_{t^+}(x+1) = 0$ (resp. $\eta_{t^+}(x) = 0, \eta_{t^+}(x+1) = 1$); otherwise, nothing happens. Note that for the moment the particles are assumed to be identical.

To define our non-linear dynamics, for fixed $K > 0$ let $I_-^K := \{1, \dots, K\}$ and $I_+^K = \{N-1-K, \dots, N-1\}$. Now we will also associate a collection of Poisson clocks, $\{N_{0,j}(t)\}_{j \in I_-^K}$ for the left and $\{N_{j,N}(t)\}_{j \in I_+^K}$ for the right, to bonds of the form $\{0, j\}$ with $j \in I_-^K$ for the left, and $\{j, N\}$ with $j \in I_+^K$ for the right. These clocks define when a particle may be injected into our system and when a particle may be removed. A particle may enter or leave the system only in the windows $I_\pm^K$. Informally, looking at the left reservoir, a particle enters to the first free site (say this site is j, then the "in rate" is α_j), and may only leave from the first occupied site (if this is the site j, the "out rate" is γ_j). At the right boundary we have a very similar dynamics as the one just described, but instead of α_j, γ_j we will have β_j and δ_j, respectively. Note that the rates at which a particle may enter into the system or leave may depend on their position. Formally, each Poisson clock $N_{0,j}(t)$ has parameter $\eta(1) \cdots \eta(j-1)\alpha_j(1-\eta(j)) + (1-\eta(1)) \cdots (1-\eta(j-1))\gamma_j \eta(j)$, and the clocks $N_{j,N}(t)$ have parameter $(1-\eta(N-j))\beta_{N-j}\eta(N-1-j) \dots \eta(N-1) + \eta(N-j)\delta_{N-j}(1-\eta(N-1-j)) \dots (1-\eta(N-1))$. We use the term *slow reservoir* because the will multiply the rates by a factor $\kappa N^{-\theta}$. In this way, adjusting the parameters θ, κ, the reservoirs are faster or slower $-$ which reflects on the boundary conditions that we will derive in the sequel. We programmed a *Mathematica* routine to simulate exactly our dynamics. The script can be found in ***here***.

3.1.2 Generator construction

As already mentioned, the exclusion process in contact with stochastic reservoirs is a *Markov process*, that we denote by $\{\eta_t : t \geq 0\}$, which has state space $\{0,1\}^{\Lambda_N}$. The infinitesimal generator is defined as

$$\mathcal{L}_N = \mathcal{L}_{N,0} + \kappa N^{-\theta} \mathcal{L}_{N,b}, \tag{3.1.1}$$

where $\mathcal{L}_{N,0}$ corresponds to the *bulk* dynamics, and $\mathcal{L}_{N,b}$ to the *boundary* dynamics. Before fully defining the generator, we will introduce some notation that will simplify the exposition. Let $I_-^K(x) := \{1, \dots, x\} \cap I_-^K$ and $I_+^K(x) := \{x, \dots, N-1\} \cap I_+^K$. For $g : \Lambda_N \to \mathbb{R}$ define

$$(\tau_x^\pm)(g) := \prod_{y \in I_\pm^K(x)} g(y). \tag{3.1.2}$$

Moreover, let $\beta_{N-x} \equiv \beta_x$ and $\delta_{N-x} \equiv \delta_x$. For a better exposition, we will decompose the boundary generator in two terms, corresponding to the linear, $\mathcal{L}_{N,b}^L$, and non-linear, $\mathcal{L}_{L,b}^{NL}$, dynamics at the boundaries. In this way, we define $\mathcal{L}_{N,b}^L := \mathcal{L}_{N,-}^L + \mathcal{L}_{N,+}^L$, where the underscript $-, +$ corresponds to the left and right boundary, respectively. For the non-linear part we define similarly. Letting $1_x(\cdot)$ be the indicator function, with $1_x(y) = 1$ for $x = y$ and $1_x(y) = 0$ for $x \neq y$ and $x, y \in \Lambda_N$, we define the change of variables $\eta \mapsto \eta^{x,x+1}$ as

$$\eta(y)^{x,x+1} = \eta(y+1)1_x(y) + \eta(y-1)1_x(y-1) + \eta(y)(1-1_x(y))(1-1_x(y-1)), \tag{3.1.3}$$

that is, the occupation exchange $x \leftrightarrow x+1$. Clearly, if y is not one of the sites $x, x+1$ nothing changes. We also define the ocupation exchange $\eta^{(x)}(y) := 1-\eta(x)$, if $y=x$. Thus, given functions $f : \{0,1\}^{\Lambda_N} \to \mathbb{R}$ we have for the bulk

$$(\mathcal{L}_{N,0}f)(\eta) = \sum_{x=1}^{N-2} (\eta(x)(1-\eta(x+1)) + \eta(x+1)(1-\eta(x)))[f(\eta^{x,x+1}) - f(\eta)], \tag{3.1.4}$$

and for the boundary

$$\begin{aligned}
(\mathcal{L}^L_{N,-}f)(\eta) &= (\alpha_1(1-\eta(1)) + \gamma_1\eta(1))[f(\eta^{(1)}) - f(\eta)] \\
(\mathcal{L}^L_{N,+}f)(\eta) &= (\beta_1(1-\eta(N-1)) + \gamma_1\eta(N-1))[f(\eta^{(N-1)}) - f(\eta)] \\
(\mathcal{L}^{NL}_{N,-}f)(\eta) &= \sum_{x\in I_-\setminus\{1\}} (\alpha_x(1-\eta(x))(\tau^-_{x-1})(\eta) + \gamma_x\eta(x)(\tau^-_{x-1})(1-\eta))[f(\eta^{(x)}) - f(\eta)] \\
(\mathcal{L}^{NL}_{NL,+}f)(\eta) &= \sum_{x\in I_+\setminus\{N-1\}} (\beta_x(1-\eta(x))(\tau^+_{x+1})(\eta) + \delta_x\eta(x)(\tau^+_{x+1})(1-\eta))[f(\eta^{(x)}) - f(\eta)],
\end{aligned} \tag{3.1.5}$$

where $(\tau^{\pm}_x)(1-\eta) \equiv \prod_{y\in I_\pm}^K (x)(1-\eta(y))$. We scale the *time* for a factor N^2, thus defining our time-scalled generator as

$$\mathcal{L} = N^2(\mathcal{L}_{N,0} + \kappa N^{-\theta}\mathcal{L}_{N,b}). \tag{3.1.6}$$

Up to our knowledge, there is no reference of this model in the literature. For $K=1$ we get the well know *SSEP with linear reservoirs*, fully studied in the *slow* setting in [3] and the *fast* setting in [12]. For $\theta = 1, \alpha_i = 0, \gamma_i = 1$ and $\beta_i = 1, \delta_i = 0$ for all $i = 1, \ldots, K$, we obtain the model in [24]. In this way, our model is an extension to both. Mathematically, as seen in (2.0.14), $N^{-\theta}\mathcal{L}_{N,b}$ can also be seen as a time scalling acting only in the reservoirs. Physically though, that does not make sense, since we are scalling the *whole process* by a factor N^2. Thus, one should see the expression above as an abuse of notation, and let the *rates* be multiplied by a $\kappa N^{-\theta}$ factor. In this way, fixed the N^2 scale, one can interpret the change in the parameter θ as the *speed* of the reservoir, or the frequency the clocks associated to nodes with the reservoir ring. Intuitively, fixed the N^2-scale, and letting $\theta \geq 0$, increasing θ makes the reservoirs *slower*, while decreasing θ makes the reservoir *faster*. As we will see, for $\theta > 1$ the reservoirs are so slow that when going to the continuous setting we cannot directly see the rates associated to the discrete system, and we get *Neumann* boundary conditions. For $\theta = 1$ the action is slow enough that the boundary conditions are linear *functions* for $K=1$, and *non-linear functions* for $K \geq 2$. For $\theta < 1$ the action is fast enough for the methodology presented in this manuscript to not work for $K \geq 2$. We believe that the boundary conditions are of *Dirichlet* type, as in the linear case $K=1$, but a formal proof is still an open problem. Intuitively, that does make sense, since the reservoirs are fast enough that the density of particles at the boundary is constant. In the literature our *fast/slow* interpretation is sometimes exchanged as *slow*↔*fast*, which also makes sense if one fixes θ first.

3.2 Heuristics for the Hydrodynamic Limit

For a better understanding of the heuristics we start by informally explaining what we mean by *Hydrodynamic Limit*. Roughly speaking, we want to show that

$$\left| \frac{1}{N} \sum_{x \in \Lambda_N} H(\tfrac{x}{N}) \eta_{N^2 t}(x) - \int_0^1 H(u)\rho_t(u) du \right| \xrightarrow[N\to\infty]{} 0 \tag{3.2.1}$$

in probability, *i.e.,* with respect to the probability induced by the process $\eta_\cdot$ with generator given by (3.1.6) described in the previous subsection, where $\rho_t(u)$ is a (weak) solution to a PDE "induced" by our model. In this section we will compute *Dynkin's martingale* (2.0.33) applied to a particular choice of the function f. The reason for this is that the expectation of Dynkin's martingale has the expression of an integral equation. When computing the aforementioned martingale, we can have some insight to what *PDE* our model induces, and under what conditions. Finished the heuristics, we will proceed to a formal proof in the following section, and present a more precise definition of the Hydrodynamic Limit. We will start with the definition of the *empirical measure*.

Definition 3.2.1 (Empirical measure/process)**.** For each $\eta \in \{0,1\}^{\Lambda_N}$ we define the *empirical measure* π^N in $[0,1]$ as

$$\pi^N(\eta, du) := \frac{1}{N-1} \sum_{x \in \Lambda_N} \eta(x) \delta_{\frac{x}{N}(du)}, \tag{3.2.2}$$

where $\delta_{\frac{x}{N}}$ is the *Dirac measure* at $\frac{x}{N}$. To study the time evolution of π^N, associated to $\{\eta_t\}_{t\geq 0}$, we define $\pi_t^N(\eta, du) := \pi^N(\eta_{N^2 t}, du)$.

Naturally, the integral of a (test) function $H : [0,1] \to \mathbb{R}$ with respect to π_t^N is written as

$$\langle \pi_t^N, H \rangle := \int_0^1 H(u) \pi_t^N(\eta, du) = \frac{1}{N-1} \sum_{x\in\Lambda_N} H(\tfrac{x}{N}) \eta_{N^2 t}(x). \tag{3.2.3}$$

Now let $\mathcal{M}$ be the set of *finite positive measures* in $[0,1]$ endowed with the *weak topology*. Then we know that if $\{\pi^N\}_{N\geq 1}, \pi \in \mathcal{M}$ then $\forall H \in C[0,1]$,

$$\pi^N \xrightarrow[N\to\infty]{w} \pi \Leftrightarrow \langle \pi^N, H\rangle \xrightarrow[N\to\infty]{} \langle \pi, H\rangle. \tag{3.2.4}$$

Let $\{\eta_t\}_{t\geq 0}$ be our *Markov process* in $\mathcal{D}_{\{0,1\}^{\Lambda_N}}[0,T]$ (note that we are letting our time interval to be compact). Thus, we can associate the *empirical measures process* $\{\pi_t^N\}_{t\in[0,T]}$ to $\mathcal{D}_{\mathcal{M}}[0,T]$. Thanks to the $\delta_{\frac{x}{N}}$ term in π_t^N there is an injection

$$\big(\{\eta_t\}_{t\geq 0} \in \mathcal{D}_{\{0,1\}^{\Lambda_N}}[0,T]\big) \hookrightarrow \big(\{\pi_t^N\}_{t\in[0,T]} \in \mathcal{D}_{\mathcal{M}}[0,T]\big) \tag{3.2.5}$$

and the *empirical process* can also be seen as a *Markov process*. Recalling the expression (2.0.19) in the previous section, in the previous display we have $S \equiv \{0,1\}^{\Lambda_N}$ and $\mathbb{B}(S) \equiv \mathcal{M}$. In this way, the empirical measure allows us to go from the *microscopic* description of the particles to the *macroscopic* description of the PDE. To simplify the exposition, and taking advantage that this section is only an *heuristic* argument for that, we will let $\pi^N(\eta, du) \equiv \frac{1}{N}\sum_{x\in\Lambda_N}\eta(x)\delta_{\frac{x}{N}}(du)$. To see what PDE our model induces, we will compute the terms in *Dynkin's formula* (2.0.33) by taking $g_t(X_t) = \langle \pi_t^N, H\rangle$ with $H \in C^2[0,1]$ and time independent, and the inner product defined as $\langle \pi_t^N, H\rangle := \int H(u)\pi_t^N(\eta, du)$, *i.e.,*

$$M_t^H := \langle \pi_t^N, H\rangle - \langle \pi_0^N, H\rangle - \int_0^t (\partial_s + \mathcal{L})\langle \pi_s^N, H\rangle ds. \tag{3.2.6}$$

When computing the generator applied to the empirical measure, we will make some useful manipulations in order to get discrete differential operators. In this way, we are able to compare Dynkin's martingale with the weak formulations derived in Section C.6. We start with some definitions:

Definition 3.2.2. The discrete *gradient* of $F : [0,1] \to \mathbb{R}$ in $\frac{x}{N}$ for $0 \le x \le N-1$ and $1 \le x \le N$, is defined as

$$\nabla_N^+ F(\tfrac{x}{N}) := N\left(F(\tfrac{x+1}{N}) - F(\tfrac{x}{N})\right) \quad \text{and} \quad \nabla_N^- F(\tfrac{x}{N}) := N\left(F(\tfrac{x}{N}) - F(\tfrac{x-1}{N})\right), \tag{3.2.7}$$

respectively, and the discrete *laplacian*: $\triangle_N F(\frac{x}{N}) = N^2\left(F(\frac{x+1}{N}) - 2F(x) + F(\frac{x-1}{N})\right)$.

For $\eta(x)$ we define the gradient and the laplacian similarly:

$$\nabla_N^+ \eta(x) := N\left(\eta(x+1) - \eta(x)\right) \quad \text{and} \quad \nabla_N^- \eta(x) := N\left(\eta(x) - \eta(x-1)\right) \tag{3.2.8}$$

for $1 \le x \le N-2$ and $2 \le x \le N-1$ respectively, and for $2 \le x \le N-2$ we define the *laplacian* as

$$\triangle_N \eta(x) = N^2\left(\eta(x+1) - 2\eta(x) + \eta(x-1)\right) \tag{3.2.9}$$

For functions of the process η, where we evaluate the gradient and laplacian in x and not in x/N, as the functional $(\tau_x^{\pm})$, the expression is analogous.

By definition,

$$\mathcal{L}\langle \pi_s^N, H\rangle = \frac{1}{N}\sum_{x\in\Lambda_N} H(\tfrac{x}{N})\mathcal{L}\eta(x) \tag{3.2.10}$$

thus, we need to compute $\mathcal{L}\eta(x)$. On (3.1.4) take $f(\eta) = \eta(x)$. From (3.1.3) a straightforward computation using Lemma C.1.1 shows that for $2 \le x \le N-2$ we have

$$\begin{aligned}\mathcal{L}_{0,N}\eta(x) &= \sum_{y\in\Lambda_N}\left(\eta(y)(1-\eta(y+1)) + \eta(y+1)(1-\eta(y))\right)(\eta^{y,y+1}(x) - \eta(x)) \\ &= \eta(x-1) - 2\eta(x) + \eta(x+1) = \tfrac{\triangle_N \eta(x)}{N^2}.\end{aligned} \tag{3.2.11}$$

For $x = 1$ we have $\mathcal{L}_{N,0}\eta(1) = \eta(2) - \eta(1)$ and for $x = N-1$, $\mathcal{L}_{0,N}\eta(N-1) = \eta(N-2) - \eta(N-1)$. Thus,

$$N^2\mathcal{L}_{0,N}\eta(x) = \triangle_N \eta_{sN^2}(x)1_{2\le x\le N-2} + N\nabla_N^+\eta(1)1_{x=1} + N\nabla_N^-\eta(N-1)1_{x=N-1}. \tag{3.2.12}$$

For the boundary generators, taking $f(\eta) = \eta(N-1)$ we have $\mathcal{L}_{N,-}^L\eta(N-1) = \beta_1(1-\eta(N-1)) - \delta_1\eta(N-1)$, and for the left, with $f(\eta) = \eta(1)$ we have $\mathcal{L}_{N,+}^L\eta(1) = \alpha_1(1-\eta(1)) - \gamma_1\eta(1)$.

Similarly, recalling (3.1.2), for the non-linear rates:

$$\begin{aligned}\mathcal{L}^{NL}_{N,+}\eta(x) &= \left[\beta_x(1-\eta(x))(\tau^+_{x+1})(\eta) - \delta_x\eta(x)(\tau^+_{x+1})(1-\eta)\right] 1_{x\in I^K_+\setminus\{1\}},\\ \mathcal{L}^{NL}_{N,-}\eta(x) &= \left[\alpha_x(1-\eta(x))(\tau^-_{x-1})(\eta) - \gamma_x\eta(x)(\tau^-_{x-1})(1-\eta)\right] 1_{x\in I^K_-\setminus\{N-1\}}.\end{aligned} \tag{3.2.13}$$

Proceeding with (3.2.10), using (C.1.1) a simple computation shows that

$$N^2\mathcal{L}_{N,0}\langle\pi^N_s, H\rangle = \langle\pi^N_s, \triangle_N H\rangle + \eta_{N^2s}(1)\nabla^+_N H(0) + \eta_{N^2s}(N-1)\nabla^-_N H(1). \tag{3.2.14}$$

For the linear terms we have

$$\tfrac{\kappa}{N^\theta}N^2\mathcal{L}^L_{N,b}\langle\pi^N, H\rangle = \tfrac{\kappa}{N^{\theta-1}}\left(H(\tfrac{1}{N})(\alpha_1 - (\alpha_1+\gamma_1)\eta_{sN^2}(1)) + H(\tfrac{N-1}{N})(\beta_1 - (\beta_1+\delta_1)\eta_{sN^2}(N-1))\right). \tag{3.2.15}$$

And for the non-linear terms:

$$\begin{aligned}\tfrac{\kappa}{N^\theta}N^2\mathcal{L}^{NL}_{N,b}\langle\pi^N, H\rangle = \tfrac{\kappa}{N^{\theta-1}} &\sum_{x\in I^K_-\setminus\{1\}} H(\tfrac{x}{N})(\alpha_x(1-\eta(x))(\tau^-_{x-1})(\eta) - \gamma_x\eta(x)(\tau^-_{x-1})(1-\eta)) +\\ &+ \tfrac{\kappa}{N^{\theta-1}}\sum_{x\in I^K_+\setminus\{N-1\}} H(\tfrac{x}{N})(\beta_x(1-\eta(x))(\tau^+_{x+1})(\eta) - \delta_x\eta(x)(\tau^+_{x+1})(1-\eta)).\end{aligned} \tag{3.2.16}$$

Fixed a measure μ^N in $\{0,1\}^{\Lambda_N}$, we define $\rho^N_s(x) := \mathbb{E}^{\mu^N}\eta(x)$. We assume that, for N large enough, $\rho^N_s(\cdot) \sim \rho_s(\cdot)$, $\mathbb{E}^{\mu^N}\tau^1_-\eta_{N^2s}(x) \sim \tau^1_-\rho^N_s(x)$ and $\rho^N_s(x) \sim \rho_s(0)\ \forall x \in I_-$ (for the right boundary is analogous), where ρ is a weak solution of the heat equation with boundary conditions as in (C.6.8), if $\theta = 1$, or as in (C.6.9), if $\theta > 1$. Assuming that we have the aforementioned assymptotics, we might proceed as follows:

$$\begin{aligned}&\forall x \in I^K_-, \quad (\tau^-_x)(\eta_{N^2s}) \mapsto \mathbb{E}^{\mu^N}(\tau^-_x)(\eta_{N^2s}) \mapsto (\tau^-_x)(\rho^N_s(x)) \mapsto (\rho_s(0))^x\\ &\forall x \in I^K_+, \quad (\tau^-_x)(1-\eta_{N^2s}) \mapsto \mathbb{E}^{\mu^N}(\tau^-_x)(1-\eta_{N^2s}) \mapsto (\tau^-_x)(1-\rho^N_s) \mapsto (1-\rho_s(0))^x.\end{aligned} \tag{3.2.17}$$

By the rationale above, the terms arising from the *SSEP* dynamics in (3.2.14), for N large enough, become equal to:

$$-\int_0^t \langle\rho_s, \triangle H\rangle ds - \int_0^t \rho_s(0)\partial_u H(0) + \rho_s(1)\partial_u H(1) ds. \tag{3.2.18}$$

For the left boundary terms, expanding $H(\frac{x}{N})$ on $\frac{1}{N}$ we have

$$H(\tfrac{x}{N}) = H(\tfrac{1}{N}) + H'(\tfrac{1}{N})(\tfrac{x-1}{N}) + \cdots = H(\tfrac{1}{N}) + \mathcal{O}(\tfrac{x-1}{N}) \tag{3.2.19}$$

for $x \in I_-$ small enough (that is, for K small enough). For $\theta > 1$ all the boundary terms vanish and we are left in (3.2.6) with

$$0 = \langle\rho_t, H\rangle - \langle\rho_0, H\rangle - \int_0^t \langle\rho_s, \triangle H\rangle ds - \int_0^t \rho_s(0)\partial_u H(0) + \rho_s(1)\partial_u H(1) ds. \tag{3.2.20}$$

We recall that Dynkin's martingale has mean zero. For $\theta = 1$, we have that (3.2.6) equals

$$\begin{aligned}
0 = \langle \rho_t, H\rangle - \langle \rho_0, H\rangle - \int_0^t \langle \rho_s, \triangle H\rangle ds \int_0^t \rho_s(0)\partial_u H(0) + \rho_s(1)\partial_x H(1) ds - \\
- \kappa \sum_{x\in I_-^K} \int_0^t H(0)\left(\alpha_x(1-\rho_s(0))\rho_s^{x-1}(0) - \gamma_x \rho_s(0)(1-\rho_s(0))^{x-1}\right) ds - \\
- \kappa \sum_{x\in I_+^K} \int_0^t H(1)\left(\beta_x(1-\rho_s(1))\rho_s^{x-1}(1) - \delta_x \rho_s(0)(1-\rho_s(1))^{x-1}\right) ds.
\end{aligned} \tag{3.2.21}$$

Showing that $\rho \in L^2(0;T,\mathcal{H}^1(0,1))$ we may conclude that $\rho : [0,T]\times[0,1] \to [0,1]$ is a weak solution of the heat equation (see Definition C.6.1) with Robin boundary conditions:

$$\partial_u \rho_t(0) = -\kappa \sum_{x\in I_-^K} \left(\alpha_x(1-\rho_t(0))\rho_t^{x-1}(0) - \gamma_x \rho_t(0)(1-\rho_t(0))^{x-1}\right), \tag{3.2.22}$$

$$\partial_u \rho_t(1) = \kappa \sum_{x\in I_+^K} \left(\beta_x(1-\rho_t(1))\rho_t^{x-1}(1) - \delta_x \rho_t(0)(1-\rho_t(1))^{x-1}\right), \tag{3.2.23}$$

which is coherent with the particular case of the *linear SSEP* ($K = 1$) and of [24], taking $\beta = 1, \alpha = 0, \kappa = j/2$. We will not adress the formal proof for general K, given that the techniques used will be the same. Instead, we will focus on the particular case $K = 2$, where correlations are already present, which is the main feature of our model. Nevertheless, we will make references through the next sections regarding the differences between general K and $K = 2$, which will be mostly a few more computations and notation. In this way, for $K = 2$, the Dynkin's martingale has the expression

$$\begin{aligned}
&\mathcal{M}_t^N(H) = \langle \pi_t^N, H\rangle - \langle \pi_0^N, H\rangle - \int_0^t \langle \pi_s^N, \Delta_N H\rangle\, ds \\
&- \int_0^t \nabla_N^+ H(0)\eta_{sN^2}(1) - \nabla_N^- H(1)\eta_{sN^2}(N-1)\, ds \\
&- \int_0^t \kappa N^{1-\theta} H(\tfrac{1}{N})(\alpha_1 - (\alpha_1+\gamma_1)\eta_{sN^2}(1)) + \kappa N^{1-\theta} H(\tfrac{N-1}{N})(\beta_1 - (\beta_1+\delta_1)\eta_{sN^2}(N-1))\, ds \\
&- \int_0^t \kappa N^{1-\theta} H(\tfrac{2}{N})(\alpha_2\eta_{sN^2}(1) - \gamma_2\eta_{sN^2}(2) - (\alpha_2-\gamma_2)\eta_{sN^2}(1)\eta_{sN^2}(2)) ds \\
&- \int_0^t \kappa N^{1-\theta} H(\tfrac{N-2}{N})(\delta_2\eta_{sN^2}(N-2) - \beta_2\eta_{sN^2}(N-1) - (\delta_2-\beta_2)\eta_{sN^2}(N-1)\eta_{sN^2}(N-2)) ds
\end{aligned} \tag{3.2.24}$$

and by the same arguments as above, this will be shown to satisfy the weak formulation, for $\theta = 1$, referenced in (C.6.8). That is, $\rho_s(u)$ is a (weak) solution to the heat equation with non-linear Robin boundary conditions. For $\theta > 1$, the already mentioned weak formulation for the heat equation with *Neumannn* boundary conditions, also referenced in (C.6.9). We end this section with some surprising observations. Looking at (3.2.24), note that for $\delta_2 = \beta_2$ and $\gamma_2 = \alpha_2$ and $K = 2$ the action of the reservoirs disappears in the continuous setting, and we have the same PDE as for $K = 1$. For the general setting, $K > 2$, another surprising condition arises. By taking $\alpha_{x+1} = \alpha_x, \gamma_{x+1} = \gamma_x$ (and similar for the right boundary), the reservoirs induces *no explicit correlations* until the end of the windows $I^K_\pm$. To see this, note that

$$\begin{aligned}(1-\eta(x))(\tau^-_{x-1})(\eta) &= -\big((\tau^-_x)(\eta)-(\tau^-_{x-1})(\eta)\big) = -\nabla_x(\tau^-_x)(\eta),\\ \eta(x)(\tau^-_{x-1})(1-\eta) &= -(1-\eta(x)-1)(\tau^-_{x-1})(1-\eta) = -\nabla_x(\tau^-_x)(1-\eta),\end{aligned} \tag{3.2.25}$$

where we are using the notation in (C.1.1). In this way, summing by parts (C.1.1) we have

$$\sum_{x\in\Lambda_N} H(\tfrac{x}{N})\alpha_x(1-\eta(x))(\tau^-_x)(\eta) = \sum_{x=2}^{K}(\tau^-_{x-1})(\eta)\nabla_x[\alpha_x H(\tfrac{x}{N})] + \alpha_1 H(\tfrac{1}{N}) - \alpha_K H(\tfrac{K}{N})(\tau^-_K)(\eta), \tag{3.2.26}$$

where we used that $(\tau^-_0)(\eta)=1$ by definition. In this way, by the product rule (C.1.1) we have

$$\nabla_x[\alpha_x H(\tfrac{x}{N})] = \alpha_x\nabla_x H(\tfrac{x}{N}) + H(\tfrac{x-1}{N})\nabla_x\alpha_x, \tag{3.2.27}$$

and the term in (3.2.26) becomes

$$\alpha_1 H(\tfrac{1}{N}) - \alpha_K H(\tfrac{K}{N})(\tau^-_K)(\eta) + \sum_{x=2}^{K}(\tau^-_{x-1})(\eta)[\alpha_x\nabla_x H(\tfrac{x}{N}) + H(\tfrac{x-1}{N})\nabla_x\alpha_x]. \tag{3.2.28}$$

By the Taylor expansion, $H(\frac{x}{N}) = H(\frac{1}{N}) + \mathcal{O}(N^{-1})$, taking the expectation and assuming that the mean of the product is the product of the mean, we have

$$\alpha_1 H(\tfrac{1}{N}) - \alpha_K H(\tfrac{K}{N})(\tau^-_K)(\rho^N) + H(\tfrac{1}{N})\sum_{x=2}^{K}(\tau^-_{x-1})(\rho^N)\nabla_x\alpha_x + \mathcal{O}(N^{-1}). \tag{3.2.29}$$

By analogous computations one gets a similar expression for the other term for the left boundary, and the terms for the right boundary.

3.3 Proof of the Hydrodynamic Limit

To show convergence of the formulation for Dynkin's martingale (3.2.24) to (C.6.7) we will use a standard methodology named *Entropy method*[1]. Before presenting the method and the main result of this chapter, we will introduce some definitions. Recalling Definition 3.2.1 (*empirical measure*), and that the probability $\mathbb{P}_{\mu^N}$ is associated to the process $\{\eta_t\}_{t\geq 0}$, we will associate a probability measure $\mathbb{Q}^N$ to π^N_t in the following definition.

Definition 3.3.1 ($\mathbb{Q}^N$-measures)**.** Recalling that given an initial measure μ^N, $\mathbb{P}_{\mu^N}$ is associated to the process $\{\eta_t\}_{t\geq 0}$, the sequence of probability measures in $\mathcal{D}_{\mathcal{M}}[0,T]$, $\{\mathbb{Q}^N\}_{N\geq 1}$ corresponds to the process $\{\pi^N_t\}_{t\geq 0}$, and is defined as the *push-forward* of $\mathbb{P}_{\mu^N}$ induced by the map

$$\big(\mathcal{D}_{\{0,1\}^{\Lambda_N}}[0,T],\mathbb{P}_{\mu^N}\big) \longrightarrow \big(\mathcal{D}_{\mathcal{M}}[0,T],\mathbb{Q}^N\big) \tag{3.3.1}$$

$$\{\eta_t\}_{t\geq 0} \mapsto \{\pi^N_t\}_{t\geq 0}. \tag{3.3.2}$$

Definition 3.3.2 (Initial profile)**.** Let $\rho_0 : [0,1] \to [0,1]$ be a measurable function. We say that a sequence of

[1]The methodology was developed by M. Z. Guo, G. C. Papanicolaou and S. R. S. Varadhan [14].

probability measures $\{\mu^N\}_{N\geq 1}$ in $\{0,1\}^{\Lambda_N}$ is *associated with the profile* $\rho_0(\cdot)$ if for any continuous function $H:[0,1]\to\mathbb{R}$ and $\forall\delta>0$ we have

$$\lim_{N\to\infty}\mu^N\left(\eta\in\{0,1\}^{\Lambda_N}:\left|\sum_{x\in\Lambda_N}\frac{1}{N-1}H(\tfrac{x}{N})\eta(x)-\int_0^1 H(u)\rho_0(u)du\right|>\delta\right)=0. \tag{3.3.3}$$

The convergence above means that the empirical measure at time $t=0$ converges in probability with respect to a fixed measure μ^N to the deterministic measure $\rho_0(u)du$, which is absolutely continuous with respect to the Lebesgue measure, and the density is the profile $\rho_0(\cdot)$.

Example 3.3.3. Let μ^N be the *Bernouli product measure*, that is, $\nu^N_{\gamma(\cdot)}(\eta:\eta(x)=1)=\gamma(\frac{x}{N})$, with γ smooth. Furthermore, consider the initial profile given by $\rho_0(u)=\gamma(u)$. Then, by *Markov's inequality* we can bound the previous probability from above by

$$\frac{1}{\delta}E_{\nu^N_{\gamma(\cdot)}}\left[\left|\sum_{x\in\Lambda_N}\frac{1}{N-1}H(\tfrac{x}{N})\eta(x)-\int_0^1 H(u)\gamma(u)du\right|\right]. \tag{3.3.4}$$

From the continuity of both H and γ, the previous expectation vanishes as $N\to\infty$.

Defined an initial profile, the *Hydrodynamic Limit* extends this micro-to-macro relation to any $t\in[0,T]$:

Theorem 3.3.4 (Hydrodynamic Limit)**.** *Let $\{\eta_t\}_{t\geq 0}$ be the process in $\{0,1\}^{\Lambda_N}$ with generator $\mathcal{L}$ defined in (3.1.6), and let $\rho_0:[0,1]\to[0,1]$ be a measurable initial profile and $\{\mu^N\}_{N\geq 1}$ a sequence of probability measures in $\{0,1\}^{\Lambda_N}$ associated with ρ_0. Then, for all $0\leq t\leq T,\delta>0$ and $H\in C[0,1]$*

$$\lim_{N\to\infty}\mathbb{P}_{\mu^N}\left(\eta_\cdot:\left|\frac{1}{N-1}\sum_{x\in\Lambda_N}H(\tfrac{x}{N})\eta_{N^2t}(x)-\int_0^t H(u)\rho_t(u)du\right|>\delta\right)=0, \tag{3.3.5}$$

where $\rho_t(u)$ is a weak solution of the heat equation with boundary conditions as in (C.6.8), if $\theta=1$, or as in (C.6.9), if $\theta>1$.

Remark 3.3.5. Note that, mathematically, the *hydrodynamic limit* is nothing more than the *Law of Large Numbers*. Also note that the above convergence can be seen as weak convergence. Thus, the function H in the statement above is *not* the test function of (C.6) and need not to be in $C^2[0,1]$.

In order to express the statement in Section 3.3.4 in terms of the measure induced by $\pi_\cdot$ in the trajectories space, we will reformulate the statement. From [17] we know that $\mathcal{M}$ endowed with the *weak topology* is metrizable (see page 50 for an example of a metric). In this way, let $\delta(\mu,\nu)$ denote such a metric for $\mu,\nu\in\mathcal{M}$. Then (3.3.5) can be reformulated as

$$\lim_{N\to\infty}\mathbb{P}_{\mu^N}\left(\eta_\cdot\in\mathcal{D}_{\{0,1\}^{\Lambda_N}}[0,T]:\delta(\pi^N_t,\rho_t(u)du)>\delta\right)=0, \tag{3.3.6}$$

and the convergence above can be interpreted as $\pi^N_t\xrightarrow[N\to\infty]{\mathbb{P}_{\mu^N}}\rho_t(u)du$. To show (3.3.4) we will follow the *entropy method*: we start by showing that the sequence of probability measures $\{\mathbb{Q}^N\}_{N\geq 1}$ is *tight*. This way, we know that exist subsequences that converge. *Assuming* uniqueness of the weak solution of PDE (C.6.1) with either Robin or Neumann boundary conditions, in Section 3.3.2, we proceed to show that the limiting point of $\{\mathbb{Q}^N\}_{N\geq 1}$ is absolutely continuous with respect to the Lebesgue measure, *i.e.,* $\pi_t^N \xrightarrow[N\to\infty]{\mathbb{P}_{\mu^N}} \rho_t(u)du$, and that $\rho_t(u)$ is a solution to the PDE's previously mentioned. This last step will be proved in two parts: first we see that the limit $\mathbb{Q}-$measure gives full weight to $\{\pi. \in \mathcal{D}_{\mathcal{M}}[0,T] : F_R = 0\}$ or $\{\pi. \in \mathcal{D}_{\mathcal{M}}[0,T] : F_N = 0\}$, if $\theta = 1$ or $\theta > 1$, respectively. (the definition for F_R and F_N can be found in (C.6.8) and (C.6.9), respectively) To see this, we use tightness and *Portmanteau's theorem* C.5.8 to move back to the discrete space and show that indeed the limit of the sequence $\{\mathbb{Q}^N\}_{N\geq 1}$ gives the full weight desired, where we have to use some replacement lemmas, shown in the Appendix A . The second part consists in finally showing that $\rho_t(u)$ is the weak solution desired$-$ *i.e.,* it lives in the *Sobolev Space* $L^2(0;T;\mathcal{H}^1(0,1))$. This will be accomplished through the named *energy estimate*, in the Appendix B.

3.3.1 Tightness

To show the tightness of $\{\mathbb{Q}^N\}_{N\geq 1}$ we will use *Aldous conditions* in Theorem C.5.5.

Proposition 3.3.6. *The sequence $\{\mathbb{Q}^N\}_{N\geq 1}$ is tight under the Skorohod topology of $\mathcal{D}_{\mathcal{M}}[0,T]$.*

Proof. By [17] (chapter 4) we know that $\{\mathbb{Q}^N\}_{N\geq 1}$ is *relatively compact* if $\{\mathbb{Q}^{N,H}\}_N$ is *relatively compact* on $\mathcal{D}_{\mathbb{R}}[0,T]$, where $H \in C^2[0,1]$ and $\mathbb{Q}^{N,H}$ is the probability measure induced by the map

$$(\mathcal{D}_{\mathcal{M}}[0,T],\mathbb{Q}^N) \ni \pi_\cdot^N \mapsto \psi(\pi_\cdot^N) = \langle \pi_\cdot^N, H\rangle \in (\mathcal{D}_{\mathbb{R}}[0,T],\mathbb{Q}^{N,H}). \tag{3.3.7}$$

Now we proceed to show the Aldous' conditions.:

- $\forall t \in [0,T], \epsilon > 0\,, \exists K_t(\epsilon)\ \subset \mathcal{S}$ compact such that $\sup_{N\geq 1}\mathbb{Q}^{N,H}\left(\langle \pi_\cdot^N, H\rangle \in \mathcal{D}_{\mathbb{R}}[0,T] : \langle \pi_t^N, H\rangle \notin K_t(\epsilon)\right) < \epsilon$.

Take $H \in C[0,1]$ and $\epsilon > 0$. Then

$$\left|\langle \pi_t^N, H\rangle\right| = \left|\frac{1}{N}\sum_{x\in\Lambda_N} H(\tfrac{x}{N})\eta_t(x))\right| \leq \sup_{u\in[0,1]}|H(u)| = \|H\|_\infty. \tag{3.3.8}$$

Choose $K_t(\epsilon) = \overline{B_r(0)}: \quad r > \|H\|_\infty$. Then clearly

$$\mathbb{Q}^{N,H}\left(\langle \pi_\cdot^N, H\rangle \in \mathcal{D}_{\mathbb{R}}[0,T] : \langle \pi_t^N, H\rangle \notin \overline{B_r(0)}\right) = 0 < \epsilon. \tag{3.3.9}$$

- $\lim_{\gamma\to 0}\limsup_{N\to\infty}\sup_{\tau\in\mathcal{T}_N,\theta\leq\gamma}\mathbb{Q}\left(\langle \pi_\cdot^N, H\rangle \in \mathcal{D}_{\mathcal{R}}[0,T] : \left|\langle \pi_{\tau+\lambda}^N, H\rangle - \langle \pi_\tau^N, H\rangle\right| > \epsilon\right) = 0.$

Note that by definition of $\mathbb{Q}^{N,H}$:

$$\begin{aligned}
&\mathbb{Q}^{N,H}\left(\langle \pi_\cdot^N, H\rangle \in \mathcal{D}_{\mathbb{R}}[0,T] : \left|\langle \pi_{\tau+\lambda}^N, H\rangle - \langle \pi_\tau^N, H\rangle\right| > \epsilon\right)\\
&= \mathbb{Q}^N\left(\pi_\cdot^N \in \mathcal{D}_{\mathcal{M}}[0,T] : \left|\langle \pi_{\tau+\lambda}^N, H\rangle - \langle \pi_\tau^N, H\rangle\right| > \epsilon\right)\\
&= \mathbb{P}_{\mu^N}\left(\eta_\cdot \in \mathcal{D}_{\{0,1\}^{\Lambda_N}}[0,T] : \left|\langle \pi^N(\eta_{\tau+\lambda}), H\rangle - \langle \pi^N(\eta_\tau), H\rangle\right| > \epsilon\right).
\end{aligned} \tag{3.3.10}$$

Fixed H, and recalling that M_t^H is a martingal with respect to the natural filtration of η, summing and subtracting the appropriate terms we have that

$$\langle\pi^N_\tau, H\rangle - \langle\pi^N_{\tau+\lambda}, H\rangle = M^{N,H}_\tau - M^{N,H}_{\tau+\lambda} - \int_\tau^{\tau+\lambda} N^2\mathcal{L}_N\langle\pi^N_s, H\rangle ds. \tag{3.3.11}$$

And we can bound the last term in (3.3.10) as follows

$$\begin{aligned}
&\mathbb{P}^N_\mu\left(\eta_\cdot \in \mathcal{D}_{\{0,1\}^{\Lambda_N}}[0,T] : \left|\langle\pi^N_{\tau+\lambda}, H\rangle - \langle\pi^N_\tau, H\rangle\right| > \epsilon\right)\\
&\leq \mathbb{P}^N_\mu\left(\eta_\cdot \in \mathcal{D}_{\{0,1\}^{\Lambda_N}}[0,T] : \left|\int_\tau^{\tau+\lambda} N^2\mathcal{L}_N\langle\pi^N_s, H\rangle ds\right| + \left|M^{N,H}_\tau - M^{N,H}_{\tau+\lambda}\right| > \epsilon\right)\\
&\leq \mathbb{P}^N_\mu\left(\eta_\cdot \in \mathcal{D}_{\{0,1\}^{\Lambda_N}}[0,T] : \left|\int_\tau^{\tau+\lambda} N^2\mathcal{L}_N\langle\pi^N_s, H\rangle ds\right| > \epsilon/2\right) +\\
&+ \mathbb{P}^N_\mu\left(\eta_\cdot \in \mathcal{D}_{\{0,1\}^{\Lambda_N}}[0,T] : \left|M^{N,H}_\tau - M^{N,H}_{\tau+\lambda}\right| > \epsilon/2\right).
\end{aligned} \tag{3.3.12}$$

Applying Chebyshev's inequality in the term on the third line of the previous display and Markov's inequality on the fourth line, (3.3.12) is bounded from above by:

$$\leq \frac{1}{\epsilon}\mathbb{E}_{\mu^N}\left[\left|\int_\tau^{\tau+\lambda} N^2\mathcal{L}_N\langle\pi^N_s, H\rangle ds\right|\right] + \frac{1}{\epsilon^2}\mathbb{E}_{\mu^N}\left[\left(M^{N,H}_\tau - M^{N,H}_{\tau+\lambda}\right)^2\right]. \tag{3.3.13}$$

Now we work with the first term in the previous sum. Note that $\left|\triangle_N H(\frac{x}{N})\right| \leq 2\|H''\|_\infty$ and $\left|\nabla^\pm_N H(\frac{x}{N})\right| \leq \|H'\|_\infty$, where we used that $H \in C^2[0,1]$. Furthermore, note that $\left|\langle\pi^N_t, H\rangle\right| \leq \|H\|_\infty$. Recalling the computations in Section 3.2, and bounding every $\eta(y)$ that appears there by 1, one can easily check that there exists constants such that

$$\int_\tau^{\tau+\lambda} N^2\mathcal{L}_n\langle\pi^N_s, H\rangle ds \lesssim \int_\tau^{\tau+\lambda} \tfrac{1}{N^{\theta-1}}\|H'\|_\infty + \tfrac{1}{N^{\theta-1}}\|H''\|_\infty ds, \tag{3.3.14}$$

where the notation $\lesssim$ means "less than a constant times" (also refered as *approximately less than*). In this way, we clearly have for $\theta \geq 1$

$$\lim_{\gamma\to\infty}\limsup_{N\to\infty}\sup_{\tau\in\mathcal{T}_T,\lambda\leq\gamma}\mathbb{E}_{\mu^N}\left[\left|\int_\tau^{\tau+\lambda} N^2\mathcal{L}_N\langle\pi^N_s, H\rangle ds\right|\right] = 0. \tag{3.3.15}$$

Now we need to show that

$$\lim_{\gamma\to\infty}\limsup_{N\to\infty}\sup_{\tau\in\mathcal{T}_T,\lambda\leq\gamma}\mathbb{E}_{\mu^N}\left[\left(M^{N,H}_\tau - M^{N,H}_{\tau+\lambda}\right)^2\right] = 0. \tag{3.3.16}$$

From stochastic calculus $(M^{N,H}_t)^2 - [M^{N,H}]_t$ is a martingale with respect to the natural filtration $\mathcal{F}^\eta_t$ (2.0.33) (see [?] for a reference). The trick here is to write the expression (3.3.16) as a function of the quadratic variation and then bound it by a constant, similarly to what we have just done. Also from (3.2.24) we have that $[M^F]_t := \int_0^t B^F_s ds$ is the quadratic variation of M^F_t, and that $(M^F_t)^2 - [M^F]_t$ is a mean zero martingale (2.0.16). Thus, let

$$B^{N,H}_s := N^2\left(\mathcal{L}_N\langle\pi^N(\eta_s), H\rangle^2 - 2\langle\pi^N(\eta_s), H\rangle\mathcal{L}_N\langle\pi^N(\eta_s), H\rangle\right) \tag{3.3.17}$$

where $B_s^{H,N} := B_{s,-}^{H,N} + B_{s,0}^{H,N} + B_{s,+}^{H,N}$, each term corresponding to $\mathcal{L}_{N,-}, \mathcal{L}_{N,0}, \mathcal{L}_{N,+}$, respectively. This way, we have that

$$\mathbb{E}_{\mu^N}\left[\left(M_\tau^{N,H} - M_{\tau+\lambda}^{N,H}\right)^2\right] = \mathbb{E}_{\mu^N}\left[\int_\tau^{\tau+\lambda} B_s^{N,H} ds\right]. \tag{3.3.18}$$

Now we prooceed to bound $B_s^{H,N}$. The contribution from the bulk dynamics can be bounded as follows:

$$\begin{aligned} B_{s,0}^{N,H} &= N^2 \sum_{x\in\{1,N-2\}} \left(\langle \pi^N(\eta_s^{x,x+1}), H\rangle - \langle \pi^N(\eta_s), H\rangle\right)^2 \\ &= \sum_{x\in\{1,N-2\}} (\eta_s(x) - \eta_s(x+1))^2 (H(\tfrac{x+1}{N}) - H(\tfrac{x}{N}))^2 \leq \tfrac{N-1}{N^2} \left\|(H')^2\right\|_\infty. \end{aligned} \tag{3.3.19}$$

We will only make the computations for the left boundary, since for the right it is analogous. Bounding the rates in the generator by a constant, we have

$$\begin{aligned} B_{s,-}^{N,H} &\lesssim \kappa \frac{N^2}{N^\theta} \sum_{x\in I^-} \Big((\langle \pi^N(\eta_s^{(x)}), H\rangle^2 - \langle \pi^N(\eta_s), H\rangle^2) - \\ &\quad -2\langle \pi^N(\eta_s), H\rangle (\langle \pi^N(\eta_s^{(x)}), H\rangle^2 - \langle \pi^N(\eta_s), H\rangle^2)\Big) \\ &= \kappa \frac{N^2}{N^\theta} \sum_{x\in I^-} \Big(\langle \pi^N(\eta_s^{(x)}), H\rangle - \langle \pi^N(\eta_s), H\rangle\Big)^2 \leq c \frac{\kappa}{N^{\theta-1}} \|H\|_\infty^2. \end{aligned} \tag{3.3.20}$$

Analagoulsy, we have that $B_{s,+}^{N,H} \lesssim \frac{\kappa}{N^{\theta-1}} \|H\|_\infty^2$. and we have

$$\lim_{\gamma\to\infty} \limsup_{N\to\infty} \sup_{\tau\in\mathcal{T}_T, \lambda\leq\gamma} \mathbb{E}_{\mu^N} \int_\tau^{\tau+\lambda} B_s^{N,H} ds = 0. \tag{3.3.21}$$

We conclude that $\{\mathbb{Q}^{N,H}\}_{N\geq 1}$ is tight $\forall H \in C^2[0,1]$, and by Theorem C.5.5 we conclude that $\{\mathbb{Q}^N\}_{N\geq1}$ is tight in $\mathcal{D}_\mathcal{M}[0,T]$. ☐

3.3.2 Characterization of limit points

Now that we know that $\{\mathbb{Q}^N\}_{N\geq1}$ has limit points, in this subsection we will characterize them. We will start by showing that every limit point is concentrated on absolutely continuous trajectories with respect to the Lebesgue measure (that is, $\pi_t^N(du) \longrightarrow \pi_t(du) = \rho_t(u)du$). To see this, we will apply Portmanteau's theorem, and the following lemma, that can be found on most measure theory books.

Lemma 3.3.7. *If a measure μ is such that $\forall G \in C[0,1]$ we have $|\langle\mu, G\rangle| \leq \int_0^t |G(u)|du$, then μ is absolutely continuous with respect to the Lebesgue measure.*

After that, we will show that the $\mathbb{Q}-$measure gives full weight to trajectories $\pi_\cdot \in \mathcal{D}_\mathcal{M}[0,T]$, where $\rho_t(u)$ is weak solution to the heat equation with either Robin or Neumann boundary conditions, depending on the value of θ.

Proposition 3.3.8. *Let $\lim_{N\to\infty}\{\mathbb{Q}^N\}_{N\geq1} = \mathbb{Q}$. Then $\mathbb{Q}$ is concentrated on trajectories of measures absolutely continuous with respect to the Lebesgue measure.*

Proof. For fixed $H \in C[0,1]$, define the map $\pi_\cdot^N \mapsto \Theta(\pi_\cdot^N) = \sup_{0\le t\le T} \left|\langle \pi_t^N, H\rangle\right|$. Bounding $\eta_t(x) \le 1$ we have

$$\Theta(\pi_\cdot^N) \le \frac{1}{N}\sum_{x\in\Lambda_N} \left|H(\tfrac{x}{N})\right| \Leftrightarrow \mathbb{Q}^N\left(\pi_\cdot \in \mathcal{D}_{\mathcal{M}}[0,T] : \Theta(\pi_\cdot) \le \frac{1}{N}\sum_{x\in\Lambda_N}\left|H(\tfrac{x}{N})\right|\right) = 1. \tag{3.3.22}$$

By continuity of H, we have for all $\epsilon > 0$, that there exists $N \in \mathbb{N}$ such that $\forall n > N$,

$$\left|\sum_{x\in\Lambda_N}\left|H(\tfrac{x}{N})\right| - \int_0^1 |H(u)|du\right| < \epsilon. \tag{3.3.23}$$

Thus we can write

$$\mathbb{Q}^N\left(\pi_\cdot \in \mathcal{D}_{\{0,1\}^{\Lambda_N}}[0,T] : \Theta(\pi_\cdot) \le \int_0^t |H(u)|du + \epsilon\right) = 1. \tag{3.3.24}$$

If we show that for fixed ϵ the set $A_\epsilon := \{\pi_\cdot \in \mathcal{D}_{\{0,1\}^{\Lambda_N}}[0,T] : \Theta(\pi_\cdot) \le \int_0^t |H(u)|du+\epsilon\}$ is closed with respect to the *Skorohod* topology, then we can apply *Portmanteau's theorem* C.5.8 to get $\mathbb{Q}(A_\epsilon) \ge \limsup_{N\to\infty}\mathbb{Q}^N(A_\epsilon) = 1$, which clearly implies that $\mathbb{Q}(A_\epsilon) = 1$. To check that A_ϵ is closed we will show that any sequence in A_ϵ has limit in A_ϵ. This way, let $\pi_\cdot^N \xrightarrow{N\to\infty} \pi_\cdot$ in the *Skorohod topology*, where $\{\pi_\cdot^N\}_{N\ge1} \in A_\epsilon$ and $\pi_\cdot \in \mathcal{D}_{\mathcal{M}}[0,T]$.

In particular, by [17] we have that $\forall s < T \quad \pi_\cdot^N \xrightarrow{N\to\infty} \pi_\cdot \Rightarrow \pi_s^N \xrightarrow{N\to\infty} \pi_s$. Taking a sequence $(t_k)_k \searrow t$ such that $\forall k \ge 1 \quad \pi_{t_k}^N \xrightarrow{N\to\infty} \pi_{t_k}$ we have

$$\epsilon + \int_0^1 |H(u)|du \ge |\langle \pi_{t_k}, H\rangle| \xrightarrow{k\to\infty} |\langle \pi_t, H\rangle|. \tag{3.3.25}$$

Thus, A_ϵ is closed, and by *Portmanteau's theorem*, $\mathbb{Q}(A_\epsilon) = 1$. □

Theorem 3.3.9. *Let $\mathbb{Q}$ be a limit point of $\{\mathbb{Q}^N\}_{N\ge1}$, whose existence follows from the fact that the sequence $\{\mathbb{Q}^N\}_{N\ge1}$ is tight. Then we have*

$$\mathbb{Q}\left(\pi_\cdot \in \mathcal{D}_{\mathcal{M}}[0,T] : F_\theta = 0\right) = 1 \tag{3.3.26}$$

where F_θ is given in (C.6.8) for $\theta = 1$ and (C.6.9) for $\theta > 1$.

Proof. Recall that we already showed that the limit point of π_t^N is absolutely continuous with respect to the Lebesgue measure, that is $\pi_t(du) = \rho_t(u)du$. The idea to show (3.3.26) is the following: we will use *Portmanteau's theorem* (after a technicality, where we will use that $\rho_t(u)$ lives in $L^2(0,T;\mathcal{H}^1(0,1))$, where $\mathcal{H}^1(0,1)$ is the *Sobolev Space* on $[0,1]$; see Definition B.0.2), to work with the measure $\mathbb{Q}^N$. Then we will take advantage of the Replacement Lemmas A.0.6 and A.0.5 to exchange $\eta_{N^2s}(1)$ by its average; $\eta(2)$ by $\eta(1)$, and then by its average again. Afterwards, we use Dynkin's martingale and Doob's inequality (C.4.3) to show the correct convergence.

We start with the case $\theta = 1$. It is enough to show that $\forall \delta > 0$:

$$
\begin{aligned}
&\mathbb{Q}\big(\pi_\cdot \in \mathcal{D}_{\mathcal{M}}[0,t] : \rho \in L^2(0;T,\mathcal{H}^1(0,1)), \pi_t(u) = \rho_t(u)du \mid \\
&\sup_{0\le t\le T} \mid \langle \rho_t, H_t\rangle - \langle \rho_0, H_0\rangle + \int_0^t \langle \rho_s, (\partial_s + \triangle) H_s\rangle\, ds - \\
&+ \int_0^t \Big\{\rho_s(1)\partial_u H_s(1) - \rho_s(0)\partial_u H_s(0)\Big\}\, ds \\
&- \kappa \int_0^t \Big\{ H_s(1)(\beta_1 - (\beta_1+\delta_1)\rho_s(1) + (\delta_2 - \beta_2)(\rho_s^2(1) - \rho_s(1))) \Big\} ds \\
&-\kappa \int_0^t \Big\{ H_s(0)(\alpha_1 - (\alpha_1+\gamma_1)\rho_s(0) + (\gamma_2 - \alpha_2)(\rho_t^2(0) - \rho_t(0)))\Big\} ds \mid > \delta\Big).
\end{aligned}
\tag{3.3.27}
$$

The condition $\rho \in L^2(0;T,\mathcal{H}^1(0,1))$ will be shown on the next section, and we will assume at the moment to hold. For simplicity, we will take H to be time independent, but we remark that the arguments for H time dependent are the same. We note that we cannot apply *Portmanteau's theorem* directly. From [10] we know that the maps $\pi_\cdot \mapsto \int_0^T \langle \pi_s, H_1(s)\rangle ds$ and $\pi_\cdot \mapsto \sup_{0\le t\le T} \left|\langle \pi_t, H_2(t)\rangle - \langle \pi_0, H_3(0)\rangle + \int_0^t \langle \pi_s, H_4(s)\rangle ds\right|$, for any $H_i \in C[0,1]$ with $i = 1,2,3,4$, are continuous with respect to the Skohorod topology. In this way, the problem lies with the terms coming from the boundary conditions, thus making the set inside the probability above not an open set in the Skorohod space. To solve this problem, we take the following functions:

$$
\overleftarrow{\iota}^{u}_{\epsilon}(v) = \frac{1}{\epsilon} 1_{(u-\epsilon,u]}(v) \quad \text{and} \quad \overrightarrow{\iota}^{u}_{\epsilon}(v) = \frac{1}{\epsilon} 1_{[u,u+\epsilon)}(v),
\tag{3.3.28}
$$

and we define the inner product as

$$
\langle \pi_s, \overleftarrow{\iota^u_\epsilon}\rangle = \frac{1}{\epsilon}\int_{u-\epsilon}^{u} \rho_s(v)dv \quad \text{and} \quad \langle \pi_s, \overrightarrow{\iota^u_\epsilon}\rangle = \frac{1}{\epsilon}\int_{u}^{u+\epsilon} \rho_s(v)dv.
\tag{3.3.29}
$$

To not overload the notation, we will omit the conditions $\rho \in L^2(0;T,\mathcal{H}^1(0,1)), \pi_t(u) = \rho_t(u)du$ from (3.3.27), since it is clear from the context. Now we "replace" the terms that are not continuous to get

$$
\begin{aligned}
&\mathbb{Q}\Bigg(\sup_{0\le t\le T} \mid \langle \rho_t, H\rangle - \langle \rho_0, H\rangle - \int_0^t \langle \rho_s, \Delta H\rangle\, ds + \\
&+ \int_0^t \Big\{\langle \pi_s, \overleftarrow{\iota^1_\epsilon}\rangle \partial_u H(1) - \langle \pi_s, \overrightarrow{\iota^0_\epsilon}\rangle \partial_u H(0)\Big\}\, ds - \\
&- \kappa \int_0^t \Big\{ H(1)(\beta_1 - (\beta_1+\delta_1)\langle \pi_s, \overleftarrow{\iota^1_\epsilon}\rangle + (\delta_2 - \beta_2)\langle \pi_s, \overleftarrow{\iota^1_\epsilon}\rangle(\langle \pi_s, \overleftarrow{\iota^{1-\epsilon}_\epsilon}\rangle - 1) \Big\} ds - \\
&-\kappa \int_0^t \Big\{ H(0)(\alpha_1 - (\alpha_1+\gamma_1)\langle \pi_s, \overrightarrow{\iota^0_\epsilon}\rangle + (\gamma_2 - \alpha_2)\langle \pi_s, \overrightarrow{\iota^0_\epsilon}\rangle(\langle \pi_s, \overrightarrow{\iota^\epsilon_\epsilon}\rangle - 1)\Big\} ds \mid > \delta'\Bigg)
\end{aligned}
\tag{3.3.30}
$$

plus the sum of the following terms

$$
\begin{aligned}
&\mathbb{Q}\left(\pi.: \sup_{0\le t\le T} \mid \int_0^t \{(\rho_s(1) - \langle \pi_s, \overleftarrow{\iota_\epsilon^1}\rangle)\partial_u H(1)\} \mid > \delta'\right),\\
&\mathbb{Q}\left(\pi.: \sup_{0\le t\le T} \mid \int_0^t \{(\rho_s(0) - \langle \pi_s, \overrightarrow{\iota_\epsilon^0}\rangle)\partial_u H(0)\} ds \mid > \delta'\right),\\
&\mathbb{Q}\left(\pi.: \sup_{0\le t\le T} \mid \kappa\int_0^t \{H(1)((\beta_1+\delta_1)(\rho_s(1) - \langle \pi_s, \overleftarrow{\iota_\epsilon^1}\rangle))\} ds \mid > \delta'\right),\\
&\mathbb{Q}\left(\pi.: \sup_{0\le t\le T} \mid \kappa\int_0^t \{H(1)(\delta_2-\beta_2)(\rho_s(1) - \langle \pi_s, \overleftarrow{\iota_\epsilon^1}\rangle)(\rho_s(1) - \langle \pi_s, \overleftarrow{\iota_\epsilon^{1-\epsilon}}\rangle)\} ds \mid > \delta'\right),\\
&\mathbb{Q}\left(\pi.: \sup_{0\le t\le T} \mid \kappa\int_0^t \{H(1)((\delta_2-\beta_2)(\rho_s(1) - \langle \pi_s, \overleftarrow{\iota_\epsilon^1}\rangle)\langle \pi_s, \overleftarrow{\iota_\epsilon^{1-\epsilon}}\rangle)\} ds \mid > \delta'\right),\\
&\mathbb{Q}\left(\pi.: \sup_{0\le t\le T} \mid \kappa\int_0^t \{H(1)((\delta_2-\beta_2)(\rho_s(1) - \langle \pi_s, \overleftarrow{\iota_\epsilon^{1-\epsilon}}\rangle)\langle \pi_s, \overleftarrow{\iota_\epsilon^1}\rangle)\} ds \mid > \delta'\right),\\
&\mathbb{Q}\left(\pi.: \sup_{0\le t\le T} \mid \kappa\int_0^t (\delta_2-\beta_2)\{H(1)(\rho_s(1) - \langle \pi_s, \overleftarrow{\iota_\epsilon^1}\rangle)\} ds \mid > \delta'\right),
\end{aligned}
\tag{3.3.31}
$$

plus the terms from the left boundary. To see that all the terms above vanish in the limit, we show that

$$
\left|\rho_s(u) - \langle \pi_s, \overleftarrow{\iota}_\epsilon^u\rangle\right| \le \frac{1}{2}\epsilon\|\partial_u\rho\|_2^2, \tag{3.3.32}
$$

as follows. Since for now we are assuming that $\rho \in L^2(0,T;\mathcal{H}^1(0,1))$, the norm above is finite and we have for $u \in [0,1]$

$$
\frac{1}{\epsilon}\int_0^\epsilon \rho_s(0) - \rho_s(u) du = -\frac{1}{\epsilon}\int_0^\epsilon\int_0^u \partial_u\rho_s(v) dv du. \tag{3.3.33}
$$

By *Cauchy-Schwarz's inequality* we have that the absolute value of the previous expression is bounded from above by

$$
\left|\frac{1}{\epsilon}\int_0^\epsilon \left[\int_0^u (\partial_u\rho_s(v))^2 dv \int_0^u 1 dv\right] du\right| = \left|\frac{1}{\epsilon}\int_0^\epsilon \left(u\int_0^u (\partial_u\rho_s(v))^2 dv\right) du\right|. \tag{3.3.34}
$$

Since $\int_0^u(\partial_u\rho_s(v))^2 dv \le \|\partial_u\rho\|_2^2 < \infty$, we can bound the previous expression by

$$
\frac{1}{\epsilon}\|\partial_u\rho\|_2^2\int_0^\epsilon u du = \frac{1}{2}\epsilon\|\partial_u\rho\|_2^2 \xrightarrow{\epsilon\to 0} 0. \tag{3.3.35}
$$

The same bound holds for $\langle \pi_s, \overrightarrow{\iota}_\epsilon^u\rangle$.

Remark 3.3.10. For the general case $K \ge 2$, the main problem are the terms of the form $\rho_s^{K-1}(0)$ and $(1-\rho_s(0))^{K-1}$ (and similar for the right boundary). A simple way to solve this is to proceed by induction.

Since $a^2 = (a+b_1-b_1)(a+b_2-b_2) = (a-b_1)(a-b_2) + b_1(a-b_1) + b_2(a-b_2) + b_1b_2$ and we have that $b_1b_2a = b_1b_2(a+b_3-b_3) = b_1b_2(a-b_3) + b_1b_2b_3$, taking $a \equiv \rho_s(0)$ and $b_j \equiv \langle \pi_s, \overrightarrow{\iota_\epsilon^{(j-1)\epsilon}}\rangle$ for $j \ge 0$, we can replace $\rho_s^{K-1}(0)$ by $\prod_{j=0}^{K-2}\langle \pi_s, \overrightarrow{\iota_\epsilon^{j\epsilon}}\rangle$ plus a sum of terms that vanish when $\epsilon \to 0$ in the limit. For the right boundary the argument is analogous.

To finally apply Portmanteau's theorem, we argue that we can approximate $\overleftarrow{\iota}_\epsilon^u$, $\overrightarrow{\iota}_\epsilon^u$ by continuous functions in such a way that the error vanishes as $\epsilon \to 0$. Now we apply Portmanteau's theorem and, recalling the definition of $\mathbb{Q}^N$ we bound from above (3.3.30) by

$$\begin{aligned}
&\liminf_{N\to\infty} \mathbb{P}_{\mu^N}\Big(\eta. \in \mathcal{D}_{\{0,1\}^{\Lambda_N}}[0,T] : \sup_{0\le t\le T} |\, \langle \pi_t^N, H\rangle - \langle \pi_0^N, H\rangle - \int_0^t \langle \pi_s^N, \Delta H\rangle\, ds \\
&+ \int_0^t \Big\{\langle \pi_s^N, \overleftarrow{\iota_\epsilon^1}\rangle \partial_u H(1) - \langle \pi_s^N, \overrightarrow{\iota_\epsilon^0}\rangle \partial_u H(0)\Big\}\, ds - \\
&- \kappa \int_0^t \Big\{H(1)(\beta_1 - (\beta_1+\delta_1)\langle \pi_s^N, \overleftarrow{\iota_\epsilon^1}\rangle + (\delta_2-\beta_2)\langle \pi_s^N, \overleftarrow{\iota_\epsilon^1}\rangle(\langle \pi_s^N, \overleftarrow{\iota_\epsilon^{1-\epsilon}}\rangle - 1)\Big\} ds - \\
&- \kappa \int_0^t \Big\{H(0)(\alpha_1 - (\alpha_1+\gamma_1)\langle \pi_s^N, \overrightarrow{\iota_\epsilon^0}\rangle + (\gamma_2-\alpha_2)\langle \pi_s^N, \overrightarrow{\iota_\epsilon^0}\rangle(\langle \pi_s^N, \overrightarrow{\iota_\epsilon^\epsilon}\rangle - 1)\Big\} ds \,|> \delta'\Big).
\end{aligned} \tag{3.3.36}$$

Remark 3.3.11. Note that with Portmanteau's theorem we are technically doing a discretization of the continuous version of the equation above. The heuristic comes into play in having some insight to what weak solutions we will get. This way, the computations above and below can be seen almost as "going backwards" from what we did in Section 3.2.

Summing and subtracting $\int_0^t \mathcal{L}_N \langle \pi_s^N, H\rangle ds$ in (3.3.36), and recalling our expression for the Dynkin's martingale in (3.2.24), we can bound the last probability by the sum of the following terms:

$$\begin{aligned}
&\liminf_{N\to\infty} \mathbb{P}_{\mu^N}(\sup_{0\le t\le T} \big|M_t^N\big| \ge \delta''), \\
&\liminf_{N\to\infty} \mathbb{P}_{\mu^N}(\sup_{0\le t\le T} \Big|\int_0^t \langle \pi_s^N, \triangle_N H\rangle - \langle \pi_s^N, \Delta H\rangle ds\Big| > \delta''), \\
&\liminf_{N\to\infty} \mathbb{P}_{\mu^N}(\sup_{0\le t\le T} \Big|\eta(N-1)\nabla_N^- H(1) - \langle \pi_s^N, \overleftarrow{\iota_\epsilon^1}\rangle \partial_u H(1)\Big| > \delta''), \\
&\liminf_{N\to\infty} \mathbb{P}_{\mu^N}(\sup_{0\le t\le T} \Big|\eta(1)\nabla_N^+ H(0) - \langle \pi_s^N, \overrightarrow{\iota_\epsilon^0}\rangle \partial_u H(0)\Big| > \delta''), \\
&\liminf_{N\to\infty} \mathbb{P}_{\mu^N}(\sup_{0\le t\le T} \Big|H(\tfrac{N-1}{N})(\beta_1 - (\beta_1+\delta_1)\eta_{sN^2}(N-1)) - H(1)\beta_1 - (\beta_1+\delta_1)\langle \pi_s^N, \overleftarrow{\iota_\epsilon^1}\rangle\Big| > \delta''), \\
&\liminf_{N\to\infty} \mathbb{P}_{\mu^N}(\sup_{0\le t\le T} |\, H(\tfrac{N-2}{N})(\delta_2-\beta_2)\eta_{sN^2}(N-1)\eta_{sN^2}(N-2) - \\
&- H(1)(\delta_2-\beta_2)\langle \pi_s^N, \overleftarrow{\iota_\epsilon^1}\rangle(\langle \pi_s^N, \overleftarrow{\iota_\epsilon^{1-\epsilon}}\rangle - 1) \,|> \delta''),
\end{aligned} \tag{3.3.37}$$

plus the terms from the left boundary. The second term on the last display vanishes as $N \to \infty$, since $\langle \pi_s^N, \triangle_N H\rangle - \langle \pi_s^N, \Delta H\rangle \xrightarrow{N\to\infty} 0$. To bound the first, we will use Doob's inequality. For the others we will apply the replacement lemmas as follows. Let

$$\overrightarrow{\eta}_{sN^2}^{\epsilon N}(1) := \frac{1}{\epsilon N}\sum_{x=2}^{1+\epsilon N} \eta_{sN^2}(x), \qquad \overleftarrow{\eta}_{sN^2}^{\epsilon N}(N-1) := \frac{1}{\epsilon N}\sum_{x=N-1-\epsilon N}^{N-2} \eta_{sN^2}(x) \tag{3.3.38}$$

Then, since $\overleftarrow{\eta}_{sN^2}^{\epsilon N}(N-1) = \langle \pi_s^N, \overleftarrow{\iota}_\epsilon^1\rangle$ (resp. $\overrightarrow{\eta}_{sN^2}^{\epsilon N}(1) = \langle \pi_s^N, \overrightarrow{\iota}_\epsilon^0\rangle$) we have $\overleftarrow{\eta_{sN^2}}^{\epsilon N}(N-1) \sim \rho_s(1)$ (resp $\overrightarrow{\eta_{sN^2}}^{\epsilon N}(1) \sim \rho_s(0)$) , we will show in Section (A) that we can exchange $\eta_{sN^2}(N-1)$ (resp. $\eta_{sN^2}(1)$) by the averages above. Due to the *continuum* embeding of $\{1, \dots, N-1\}$ in $[0,1]$, it is intuitive that points close enough in the discrete setting will be indistinguishable in the continuous setting. We will also quantify this in the following subsection, showing that we can indeed replace $\eta(N-2)$ by $\eta(N-1)$ (analogous for the left). Thus, we now proceed in the following way:

- For the bulk terms in (3.3.37), we apply Lemma A.0.6 with the choice $\psi(\eta)=1$ and replace $\eta_{sN^2}(N-1)$ (resp. $\eta_{sN^2}(1)$) by $\overleftarrow{\eta}^{\epsilon N}_{sN^2}(N-1)$ (resp. $\overrightarrow{\eta}^{\epsilon N}_{sN^2}(1)$), paying a price $\mathcal{O}(N^{-1})$.

- For the *linear* boundaries terms in (3.3.37), we apply again Lemma A.0.6 with $\psi(\eta)=1$.

- We treat $\eta_{sN^2}(N-2)$ (resp. $\eta_{sN^2}(2)$) by first applying Lemma A.0.5 with $\psi(\eta)=1$ to replace $\eta_{sN^2}(N-2)$ (resp. $\eta_{sN^2}(2)$) by $\eta_{sN^2}(N-1)$ (resp. $\eta_{sN^2}(1)$). Again, we apply Lemma A.0.6 with $\varphi(\eta)=\eta_{sN^2}(N-1)$ (resp. $\varphi(\eta)=\eta_{sN^2}(1)$) with a cummulative error of $\mathcal{O}(N^{-1})$.

- For the correlation terms $\eta_{sN^2}(N-1)\eta_{sN^2}(N-2)$ (resp. $\eta_{sN^2}(1)\eta_{sN^2}(2)$), we first replace $\eta_{sN^2}(N-2)$ by $\overrightarrow{\eta}^{\epsilon N}_{sN^2}(N-1)$ with the choice $\psi(\eta)=\eta_{sN^2}(N-1)$ (similar for the left). Now that we have the term $\eta_{sN^2}(N-1)\overrightarrow{\eta}_{sN^2}(N-1)^{\epsilon N}$, then we replace $\eta_{sN^2}(N-1)$ by $\overrightarrow{\eta}^{\epsilon N}_{sN^2}(N-1)$ by Lemma A.0.6 with $\psi(\eta)=\eta^{\epsilon N}_{sN^2}(N-1)$ with a cummulative error of $\mathcal{O}(N^{-1})$.

- Observing that $\langle \pi^N_s, \overrightarrow{\iota}^{\,0}_{\epsilon}\rangle = \overrightarrow{\eta}^{\epsilon N}_{sN^2}(1)$, $\langle \pi^N_s, \overleftarrow{\iota}^{\,1}_{\epsilon}\rangle = \overrightarrow{\eta}^{\epsilon N}_{sN^2}(N-1)$ and
$$\langle \pi^N_s, \overrightarrow{\iota}^{\,0}_{\epsilon}\rangle\langle \pi^N_s, \overrightarrow{\iota}^{\,\epsilon}_{\epsilon}\rangle = \overrightarrow{\eta}^{\epsilon N}_{sN^2}(1)\overleftarrow{\eta}^{\epsilon N}_{sN^2}(\epsilon N+1)+\mathcal{O}((\epsilon N)^{-1}), \tag{3.3.39}$$
we are done.

Now we are left with the contribution from the *Dynkin's martingale*, that we shall bound as follows:

$$\mathbb{P}_{\mu^N}\left(\sup_{0\le t\le T}\left|M^N_t\right|>\delta\right)\le \frac{2}{\delta}\mathbb{E}_{\mu^N}\left(\left|M^N_T\right|^2\right)^{\frac12}=\frac{2}{\delta}\mathbb{E}_{\mu^N}\left(\int_0^T B^{N,H}_s ds\right)^{\frac12} \tag{3.3.40}$$

where we applied *Doobs's inequality* C.4.3 in the first step. Recalling that $\int_0^T B^{N,H}_s ds$ is the quadratic variation of *Dynkin's martingale*, one can proceed as we did in order to show tightness and we are done.

For $\theta>1$, by the same arguments we arrive at

$$\begin{aligned}&\liminf_{N\to\infty}\mathbb{P}_{\mu^N}\Bigg(\sup_{0\le t\le T}\mid\langle\pi^N_t,H\rangle-\langle\pi^N_0,H\rangle-\int_0^t\langle\pi^N_s,\Delta H\rangle\,ds\\&+\int_0^t\{\langle\pi^N_s,\overleftarrow{\iota^1_\epsilon}\rangle\partial_uH(1)-\langle\pi^N_s,\overrightarrow{\iota^0_\epsilon}\rangle\partial_uH(0)\}\,ds\Bigg).\end{aligned} \tag{3.3.41}$$

We proceed exactly the same way, summing and subtracting $\int_0^t \mathcal{L}_N\langle\pi^N_s,H\rangle ds$ and bounding each term as in (3.3.37), and applying the Replacement Lemma A.0.6 to exchange $\eta_{sN^2}(1)$ and $\eta_{sN^2}(N-1)$ by respective boxes. Afterwards, the procedure is identical. □

Chapter 4

Propagation of Chaos

The present chapter will be dedicated to expose the main arguments of [24] to show sharp estimates for the so called $v-$functions, $v^{\epsilon}(\underline{x}, t \mid \mu^{\epsilon})$, for "small times". These $v-$functions are closely related to the correlations between particles, and to the *propagation of chaos*. In the literature, *propagation of chaos* refers to the property that, in a particle system, any finite number of particles will evolve independently as the total number of particles goes to infinity $-$ as stated in [30]. Recalling the presentation of our model in section 3.1 and subsection 3.1.2 of the previous chapter, and recalling the *master equation* (2.0.32), we see that for $K > 2$ the reservoirs induce correlations in (2.0.32), thus the equation is not *closed* in terms of $\rho^N(t, x)$. Considering the solution to the closed equation that we would have if all particles were independent of each other, the $v-$functions measure the *"closeness"*, in mean, of the system, as a whole, to the one where these correlations are absent. In this way, showing sharp bounds for the $v-$functions as a function of the number of initial particles in the system indirectly shows the propagation of chaos property, hence the name of the present chapter. For more results in this direction we refer the reader to [23].

Through this chapter, we will make the necessary adaptations for our model, which are mostly introducing the parameter θ in the right place, and doing the computations in the original article to see where this leads us to. In this way, the largest difficulty lies mostly in understanding the computations and grasping the main ideas. Given that the article if very technical, our main focus will be to explain to the reader the largest steps, ideas, difficulties and intuition for the proofs. Thus, in some proofs we will refer the reader to [24] for the full computations. Nevertheless, we will do many computations through this chapter.

Letting $\epsilon := 1/N$, the main result of this chapter is that the $v-$functions are of the order of $c_n(\epsilon^{-2}t)^{-c^*n}$, for times smaller than ϵ^{β^*}, where n is the *initial* number of particles in the system, c_n, c^* are constants (to be specified), and $\beta^* > 0$. Thus, the $v-$functions vanish as the number of particles goest to infinity. As mentioned in the previous chapter, in [24] particles could only enter from the reservoir in the *right*, and leave from the reservoir in the *left*. Plus, the "frequency" parameter θ was set to be equal to 1. Here, we show that this result holds (with a very simple modification) for our model, for $\theta \geq 1$. We conjecture that for $\theta > 1$ the correct bound is $c_n(\epsilon^{-2}t)^{-\theta c^* n}$ instead, given that the action of the reservoirs is very slow. In some instances (to be explained in section 4.5) we were able to show these estimates with slightly different arguments, but in others the main arguments in the original paper could not be adapted in order to get the desired bound. Moreover, we could not formulate new arguments for these instances. In this way, the decay $c_n(\epsilon^{-2}t)^{-\theta c^* n}$ is still a conjecture. Since we were able to show this for some cases, we believe there might be something missing, and not that the whole methodology does not work$-$ in contrast to the case $\theta < 1$, as we will see.

The main body of this chapter will be as follows. In section 4.1, we present some definitions and notation for the chapter, defining precisely what was explained above, regarding the $v-$functions. In the subsection 4.1.1, we define the exclusion process in the bulk (in the present chapter, named *stirring process*) through another process, named *Active/Passive marks process*, that registers every time a *Poisson clock rings* (as explained in section 3.1); and define a coupling between the original process and a process with *independent* particles. In section 4.2, we derive a discrete equation for v, then apply *Duhamel's formula* to get an integral equation, where we proceed to bound the transition probabilities by taking advantage of (E.1.2). We end this section by finding a series of successive bounds for v. Each time we apply one bound to a term of v, we get a new expression, again in terms of v but now with *more* or *less* particles. In this way, we see that one either applies these successive bounds indefinitely, or arrives at a step where there are no more particles left in v. Thus, in section 4 we will classify each term of these successive bounds/iterations through a process named *skeleton* and a *branching process*, that determines the instant that a particle is removed (or added) from v. If we have infinitely many iterations, we will truncate this series at a fixed step, hence the name *truncated hyerarchy*. Moreover, we will see that this hyerarchy induces a partition in our time interval. We will then *choose* an appropriate partition, and break the following proofs in two parts: when the times at each iteration are said to *cluster to* t or not (to be better explained in section 4.4). Finally, in the following section with respect to the one we just have just mentioned, we will derive bounds when these times are said to either "cluster" or not. Here, the coupling described in section 4.1 will play a major role in the proof of when there is a cluster. In the last section, 4.5, we derive the said estimates for the $v-$functions for "small" times, *i.e.,* $t \leq \epsilon^{\beta^*}$.

4.1 Notation and first definitions

Recalling the definition of our generator in section 3.1.2, we define for a function u the operators

$$D_{\pm,0}u(x) := u(x)(\tau^{\pm}_{x\pm 1})(1-u) \quad \text{and} \quad D_{\pm,1}u(x) := (1-u(x))(\tau^{\pm}_{x\pm 1})(u). \tag{4.1.1}$$

We will denote $I^K_{\pm}$ by $I_{\pm}$, since K is fixed. Moreover, following the notation in [24], we let $\epsilon := N^{-1}$. With this notation, the boundary generators applied to $f(\eta) = \eta(x)$ now take the form

$$\kappa\epsilon^{\theta}\mathcal{L}_{N,-}\eta(x) = \sum_{x\in I^K_-} \kappa\epsilon^{\theta}\alpha_i D_{-,1}\eta(x) - \kappa\epsilon^{\theta}\gamma_i D_{-,0}\eta(x) \tag{4.1.2}$$

and similarly for the right. For simplicity, we will take $\kappa = 1$ and $\alpha_i = \alpha, \beta_i = \beta, \gamma_i = \gamma, \delta_i = \delta$. We will not focus at all in these terms, and one can check that the proofs in the following sections are completely identical for any choice of parameters. Moreover, will denote the generator of the bulk dynamics, $\mathcal{L}_{N,0}$, by the generator of the *stirring process*. If we had $\perp\!\!\!\perp_{x\in I_\pm} \eta(x)$ a direct application of *Kolmogorov's equation* (2.0.32) would lead to:

$$\begin{aligned}&\frac{\partial}{\partial t}\rho_\epsilon(x,t) = \triangle_\epsilon\rho_\epsilon(x,t) + \epsilon^{\theta-2}1_{x\in I_-}\left(\alpha D_{-,1}\rho_\epsilon(x,t) - \gamma D_{-,0}\rho_\epsilon(x,t)\right)\\ &- \epsilon^{\theta-2}1_{x\in I_+}\left(\beta D_{+,1}\rho_\epsilon(x,t) - \delta D_{+,0}\rho_\epsilon(x,t)\right),\end{aligned} \tag{4.1.3}$$

with $\triangle$ defined below. In this way, refering to the initial condition as $\rho(x,0) = \mu_\epsilon[\eta(x,0) = 1]$, where μ_ϵ is a *product measure*, we shall rewrite the equation above by

$$\begin{aligned}&\frac{\partial}{\partial t}\rho_\epsilon(x,t) = \triangle_\epsilon\rho_\epsilon(x,t) + \epsilon^\theta D_\pm^\epsilon\rho_\epsilon\\ &\rho(x,0) = \mu_\epsilon[\eta(x,0) = 1].\end{aligned} \tag{4.1.4}$$

The (discrete) laplacian is now a discrete laplacian with *reflecting boundary conditions*.

$$\triangle_\epsilon u(x) := u(x+1) + u(x-1) - 2u(x), \quad \text{for} \quad 1 < x < N-1 \tag{4.1.5}$$

and for the boundaries,

$$\triangle_\epsilon u(N-1) := -(u(N-1) - u(N-2)) \quad \text{and} \quad \triangle_\epsilon u(1) := -(u(1) - u(2)), \tag{4.1.6}$$

for a function u. Letting $\Lambda_N^{n,\neq}$ with $n \geq 1$ be the set of all sequences $\underline{x} = (x_1, \ldots, x_n)$ such that $x_i \neq x_j$, we define the $v-$functions as

$$v^\epsilon(\underline{x}, t \mid \mu_\epsilon) := \mathbb{E}_\epsilon\left[\prod_{i=1}^n(\eta(x_i,t) - \rho_\epsilon(x_i,t))\right], \qquad \underline{x} \in \Lambda_N^{n,\neq}, n \geq 1 \tag{4.1.7}$$

where the process above starts with a product measure μ^ϵ, and $\rho_\epsilon(x,t)$ is solution to (4.1.3). The main result of this chapter is the following bound for the $v-$functions:

Theorem 4.1.1. *There exist $c^* > 0$ so that, $\forall\beta^* > 0$ and positive integer n, $\exists c_n < \infty$ constant such that for any $\epsilon > 0$ and initial product measure μ^ϵ, for $\theta \geq 1$ and $t \leq \epsilon^{\beta^*}$ holds*

$$\sup_{\underline{x}\in\Lambda_N^{n,\neq}} |v^\epsilon(\underline{x}, t \mid \mu^\epsilon)| \leq c_n(\epsilon^{-2}t)^{-c^* n}. \tag{4.1.8}$$

We will not show the extension for "long times" ($t \leq \tau\log\epsilon^{-1}$ for some τ), that can be found in [24]. We remark, however, that although this result is shown in [24] with a short proof, the extension for "long times" is not trivial, and an open problem for many models. For a complete discussion regarding this topic, we refer the reader to [23]. We also remark that this extension is consequence of the solution to (4.1.3), where the *Gauss Kernel* (E.1.2) plays the main role, which appears, in turn, as consequence of the exclusion process in the bulk. For the model studied through this thesis, this extension is completely analogous, and the boundary conditions do not show many technical difficulties, as opposed to the following sections.

4.1.1 Coupling and probability estimates

In this section we will start by defining a *realization* of the *stirring process* in terms of an *Active/Passive marks process* (A/P-process). This process is useful because it "saves" all the information regarding the movement of each particle, registering the action of each clock, and atributing marks to each particle. Then, we will take advantage of the *dual* process to look only at set of particles at a given time. Together with the A/P-process, given a set of particles at a fixed time, we will trace the evolution back to the initial configuration and label each individual particle. To completely define the evolution of the process, we mark the times when two particles are neighbors, and each time a *Poisson clock* rings, through stopping times and Point processes. Finally, we will couple the *stirring process* with a "similar" process, with the exception that every particle is independent of each other. In this way, for the forbidden jumps in the original process to be well associated to the coupled process, we will define these jumps in the coupled process as "colisions". In the end of this section, we will state the results, regarding the coupling, that will be more useful though this chapter.

Definition 4.1.2 (A/P-process)**.** The *A/P-process* is *realized* in a probability space (Ω, P_ϵ). It is defined as a product of *Poisson processes* indexed by $\{x, x+1\}$ with $x \in \mathbb{Z}$. For each *bond*, we associate a *Poisson process* with intensity ϵ^{-2}. Its events are named "marks". To each mark we associate (independently) an atribute (*passive* or *active*) with probability $1/2$. Each Poisson process is mutually independent, and their *common* distribution is denoted by $\mathbf{P}_\epsilon$.

For any *realization* $\omega \in \Omega$, we define the evolution in Λ_N as follows. When a particle is in a node where a *Poisson clock* rings, it moves to its neighbor site. If that site is occupied, both particles *exchange* their positions.

Recalling our exposition of the *dual process* in (2.0.43), we will denote by $X(t) \subset \Lambda_N$ the set of occupied particles at time t, *i.e.,* $x \in X(t) \Leftrightarrow \eta(x,t) = 1$. It is clear that $X(t)$ has the law of the stirring process, with generator $\epsilon^{-2}\mathcal{L}_{N,0}$. From (2.0.43), we recall that the generator of the dual process has the evolution *inverted*. For us, in practice, this means that given $\omega \in \Omega$ and $X(t)$, we follow backwards the process and then define $X(0)$. This leads to the proof of the following result, that can be found both in [17] and [22].

Proposition 4.1.3. *For any $X \subset \Lambda_N, \eta_0 \in \{0,1\}^{\Lambda_N}$ and $t > 0$ we have*

$$\mathbf{E}_\epsilon\left[\prod_{x\in X} \eta(x,t) \mid \eta(\cdot,0) = \eta_0\right] = \mathbf{E}_\epsilon\left[\prod_{x\in X(t)} \eta_0(x) \mid X(0) = X\right]. \tag{4.1.9}$$

This is useful because now we can study the *stirring process* by looking at the particles that we have at the moment. Given a realization ω, we can follow each particle and label them, as in the following definition.

Definition 4.1.4. Given $\omega \in \Omega$, denote by $\underline{x} = (x_{i_1}, \ldots, x_{i_n})$ a *labeled configuration* of n particles. The labels are the indexes i_k, while the particles are x_{i_k}. We denote by $\underline{x}(t)$ the *labeled version* of $X(t)$. Moreover, we let $\underline{x} \equiv \underline{x}(0)$ be the initial distribution of particles. Note that, while exchanging particles has no efect in $X(t)$ since $X(t)$ only recorders the occupied positions, it does in $\underline{x}(t)$.

Now that the particles are labelled, we let $x_1 \sim x_2$ mean that the particles x_1 and x_2 are neighbors (in other words, are in the same bond). We denote the event of a mark appearing between particles x_1 and x_2 by $x_1 M x_2$. Now, we define the (stopping) times associated to two neighbor particles.

Definition 4.1.5. Given the *initial position* x_i and x_j of the particles i, j, we define in Ω the random multi-interval $\mathcal{T}_{x_i \sim x_j} := \{s \geq 0 : x_i(s) \sim x_j(s)\}$, that is, the "continuous" times when they are neighbors. Restricting this to $[0, t]$, we define $\mathcal{T}_{x_i \sim x_j}(t) := \mathcal{T}_{x_i \sim x_j} \cap [0, t]$. Now we let

$$I_{x_1 \sim x_2}(t) := \{(s, y_1, y_2) : s \in \mathcal{T}_{x_i \sim x_j}(t), y_i = x_i(s), y_1 M y_2\}. \tag{4.1.10}$$

We define the stopping time $\tau_{x_1 \sim x_2}$ as $\min_s \in I_{x_1, x_2}(\infty)$. Moreover, we associate a counting process to $I_{x_1 \sim x_2}(t)$, $N_{x_1 \sim x_2}(t) := |I_{x_1 \sim x_2}(t)|$.

In order for the reader to be familiar with the labeled and the A/P process, from the definitions above we can show the following lemma.

Lemma 4.1.6. *Let $\underline{x} = (x_1, \dots, x_n), t > s > 0$ and $f(y_1, \dots, y_n)$ be an antisymmetric function under the exchange of y_1 and y_2. Then*

$$\mathbf{E}_{\epsilon, \underline{x}} \left[1_{\tau_{x_1 \sim x_2} \leq s} f(\underline{x}(t)) \right] = 0, \tag{4.1.11}$$

where $\underline{x}$ is the initial condition, that is, $\mathbf{E}_{\epsilon, \underline{x}}(\cdot) = \mathbf{E}_{\underline{x}} [\cdot \mid \underline{x}(0) = \underline{x}]$.

Proof. Given two particles x_1, x_2 and two realization of the A/P-process, $\omega, \omega' \in \Omega$, we say that ω and ω' are *similar* (and denote it by $\omega \equiv \omega'$) if all marks at all pairs $\{x, x+1\}$ occur at the same time both in ω and ω', and the *active* and *passive* attributes are the same in both, *except* for the marks in $I_{x_1 \sim x_2}(s)$ *i.e.,* when they are neighbors.

In this way, we have that $\omega \equiv \omega' \Rightarrow$ the realizations evolved from the same initial configuration. Moreover, $\forall t$ we have ω equals ω' except *at most* for an exchange of the particles with label 1 and 2. Almost by definition we see that $\equiv$ is an *equivalence* relation (hence the notation). Clearly, $\omega \equiv \omega$ and $\omega \equiv \omega' \Rightarrow \omega' \equiv \omega$. And since they differ at most in the marks $x_1 M x_2$, given another realization ω'' we have $\omega \equiv \omega', \omega' \equiv \omega'' \Rightarrow \omega \equiv \omega''$ since all three will coincide except when $x_1 M x_2$, which does not alter the random walk of each particle. Thus, notice that $N_{x_1 \sim x_2}(s)$ is *constant* in each equivalence class of a realization ω. Let $N_{x_1 \sim x_2}(s) = p$. Then, we have that $\#[\omega]_p = 2^p$, where $[\omega]_p$ denotes the equivalence class of a realization such that $N_{x_1 \sim x_2}(s) = p$, and $\#\cdot$ simply denotes the cardinality. We have 2^p elements since there are p marks and the marks of particle x_1 and x_2 may differ in each element of its class, each time there is a mark.

Each element is characterized by the active/passive attribute of the p marks in $I_{x_1 \sim x_2}(s)$. Thus, their distribution conditioned on a given class is a product of $1/2, 1/2$ probabilities (recall that the marks active/passive are attributed with probability $1/2$). In this way, with some abuse of notation, we have that $\{\tau_{x_1 \sim x_2} \leq s\} = \cup_{p \geq 1} \{N_{x_1 \sim x_2(s)} \geq p\} = \cup_{p \geq 1} \{[\omega]_p\}$. Thus, we have that Law$(\underline{x}(t) \mid \tau_{x_1 \sim x_2} \leq s)$ is symmetric under the exchange of particles $x_1(t) \leftrightarrow x_2(t)$. Since f is antisymmetric under the exchange of particles, we are done. $\square$

As mentioned in [24], in [23] and [22] it was observed that *tail estimates* prove to be good ingredients to construct good couplings. Through the present chapter, they will be most useful in section 4.4, since we will be able to relate the "distance" of the original process and the coupled process explicitly in terms of time, and we have estimates for when two particles are neighbors. Thus, we will state the following theorem.

Theorem 4.1.7. *There is c such that for all $\epsilon > 0$*

$$\sup_{x_1 \sim x_2} \mathbf{P}_\epsilon[\tau_{x_1 \sim x_2} \geq t] \leq \frac{c}{(\epsilon^{-2}t)^{1/2}+1}. \tag{4.1.12}$$

Moreover, given any $T > 0$, for any $\xi > 0$ and any $k \geq 1$ there is c such that for all $t \leq T$ and for all $\epsilon > 0$

$$\sup_{x_1, x_2} \mathbf{P}_\epsilon[N_{x_1 \sim x_2} \geq (\epsilon^{-2}t)^{1/2+\xi}] \leq c(\epsilon^{-2}t)^{-k}. \tag{4.1.13}$$

To show the theorems above, the following proposition is useful.

Proposition 4.1.8. *There is c such that for all $\epsilon > 0$*

$$\sup_{y_1, y_2} \mathbf{P}_{\epsilon, y_1, y_2}[|x_1(s) - x_2(s)| = 1] \leq \frac{c}{(\epsilon^{-2}s)^{1/2}}, \tag{4.1.14}$$

where the suffix y_1, y_2 recalls that $y_i(0) = y_i, i = 1, 2$.

Now we define the coupling.

Definition 4.1.9 (Coupling)**.** Recalling the stirring process $\underline{x}$, let $\underline{x}^0$ be a *labeled* configuration of *independent particles*. We suppose, without loss of generality, that the labels are $\{1, \ldots, n\}$, thus we let $\underline{x} = (x_1, \ldots, x_n)$ and $\underline{x}^0 = (x_1^0, \ldots, x_n^0)$. Now define:

- σ - an arbitrary *priority list*, defined as a random permutation of $\{1, \ldots, n\}$;

  A particle i has *priority* over a particle j if $\sigma(i) < \sigma(j)$.

As stated in Definition 4.1.2, given a realization of the A/P-process, we define $\underline{x}(t)$ by looking at the marks in nodes in Λ_N only. In the same space as the *A/P-process*, we define the "evolution" of $\underline{x}^0(t) = (x_1^0(t), \ldots, x_n^0(t)) \in \Lambda_N^n$, given the initial configuration $\underline{x}^0(0) := \underline{x}(0)$. The whole process $\underline{x}^0(t)$ is defined by the times at which each particle $x_i^0(\cdot)$ "tries" to jump to a neighbor site. Again, the jumps out of Λ_N are supressed. For that purpose, we define:

- $t_{i,l}$ - the times when a particle $x_i^0(\cdot)$ *tries* to move to the *left*. Similarly, we define $t_{i,r}$.

We recall that the A/P-process is defined in the whole $\mathbb{Z}$. In this way, the jumps *outside* of Λ_N are *supressed*. We remark that the times $t_{i,l}, t_{i,r}$ determine $\underline{x}^0(\cdot)$, and not the opposite, since we cannot recover from $x^0(\cdot)$ the attempted jumps that $t_{\cdot,\cdot}$ registers. In this way, since we cannot recover the times directly from $\underline{x}^0(\cdot)$, we define an auxiliary process $\underline{y}(t)$.

Definition 4.1.10 (Auxiliary process $\underline{y}(t)$)**.** Let t be the first time there is a mark in the bond $\{x, x+1\}$, and there is at least one particle in this bond, *i.e.,*

$$t := \inf\{s > 0 \mid xMx+1, \underline{x}(0) \cap \{x, x+1\} \neq \emptyset\}, \tag{4.1.15}$$

with a clear abuse of notation. We start the process $\underline{y}$ in $\underline{x}^0(0)$, that is $\forall s \in]0, t[$ set $\underline{y}(s) := \underline{x}^0(0)$. For $s \leq t$, $\underline{y}(t)$ *defines* $\underline{x}^0(s)$ as follows.

- $\underline{x}(0) \cap \{1, N-1\} = \emptyset$

  If $|\underline{x}(0) \cap \{x, x+1\}| = 1$, let us denote this particle by $x_i(0)$.

  If the mark is *passive* $\longrightarrow \underline{y}(t) = \underline{x}^0(0)$;

  If the mark is *active* $\longrightarrow y_k(t) = x_k^0(0), \forall k \neq i$ and $y_i(t) = x_i^0(0) \pm 1, x_i(0) = x$, where we have $x_i^0(0) + 1$ if $x_i(0) = x$, or $x_i^0(0) - 1$, if $x_i(0) = x + 1$;

  If $\underline{x}(0) \cap \{1, N-1\} = \{x_i(0), x_j(0)\}$. Let us take $\sigma(i) < \sigma(j)$.

  If the mark is *passive* $\longrightarrow y_k(t) = x_k^0(0), k \neq j$, and $y_j(t) = x_j^0(0) - (x_j(0) - x_i(0))$. We say that the particle j *collides* with particle i. That is, $\underline{y}(t)$ registers that the particle $x_j^0(\cdot)$ "tried" to jump, while $x_i^0(\cdot)$ stands still;

  If the mark is *active* $\longrightarrow y_k(t) = x_k^0(0), k \neq i$, and $y_i(t) = x_i^0(0) + (x_j(0) - x_i(0))$. We say that the particle i *collides* with particle j. That is, $\underline{y}(t)$ registers that the particle $x_i^0(\cdot)$ "tried" to jump, while $x_j^0(\cdot)$ stands still;

- $\underline{x}(0) \cap \{1, N-1\} \neq \emptyset$. Let $x_i(0) = 1$, *i.e.,* the first site is occupied.

  If the mark is *passive* $\longrightarrow y_k(t) = x_k^0(0), k \neq j$;

  If the mark is *active* $\longrightarrow y_k(t) = x_k^0(0), k \neq j$ and $y_i(t) = x_i^0(0) - 1$.

Note that the "particle" $y_i(t)$ actually *leaves* Λ_N. Now for the time $\underline{y}(s), \forall s \in [0, t]$, where we recall that t is the "first mark time", as in (4.1.15).

- Set $x^0(s) = \underline{y}(s) = x^0(0)$ for $s < t$, and

$$\begin{cases} \underline{x}^0(t) = \underline{y}(t), & \underline{y}(t) \in \Lambda_N^n \\ \underline{x}^0(t) = \underline{x}^0(0), & \text{otherwise} \end{cases} \tag{4.1.16}$$

  Now with $\underline{x}(t), \underline{x}^0(t)$, we define $\underline{y}(s)$ for $t < s < t_2$, where t_2 is the time of the next mark, by the same rules, inductively. We remark that with the definition above we have $\underline{y}(t^+) = \underline{x}^0(t)$. That is, while at first a particle in $\underline{y}$ may jump to outside of Λ_N, in the next instant it jumps *back* to Λ_N, registering the time at which such jump happened, *i.e.,* $\underline{y}(t^-) = \underline{y}(t) \neq \underline{y}(t^+)$. Moreover, since the jumps out of Λ_N are *supressed*, although $\underline{y}$ register its time, nothing happens in $\underline{x}^0$. Moreover, if, for example, a particle $y_i(t)$ jumped outside of Λ_N, from $x = 1$ at time t, then the two instantaneous jumps are $\{t_{i,l}, t_{i,r}\}$. Under this case, we only save the *"in"* jump, $t_{i,r}$. In this way, the process is well defined and, since we know particles only enter to Λ_N if they "instantaneously" left, both jumps can be recovered.

From [23], with the definition above, one can show many properties of this coupling, as:

- $\{t_{i,r}, t_{i,l}\}_{i=1:n}$ are mutually independent Poisson processes with rate ϵ^{-2};
- The Law of $\underline{x}^0(t)$ equals the law of n independent random walkers in Λ_N with jump rate ϵ^{-2};
- The particles x_i, x_i^0 have different jumps only when one of them is at the site 1 or $N-1$; or when there is a collision with a different particle x_j, and $\sigma(j) < \sigma(i)$, *i.e.,* x_i has a *higher* priority and "x_i collides with x_j".

- For any i and $t > 0$, $x_i(t)$ is completely determined by $y_j(s), s \in [0,t]$, where j is such that $\sigma(j) < \sigma(i)$.

- The particle with the *lowest priority* have the same walk both in the coupled process and the original process, with probability *one*, that is

$$\sigma(\ell) = 1 \wedge (x_\ell(0) = x^0_\ell(0)) \Rightarrow P(x_\ell(t) = x^0_\ell(t)) = 1, \quad \forall t \geq 0. \tag{4.1.17}$$

On this list, the last two items, together with the theorem that we state below, that quantifies the "closeness" of the stirring process to the "independent process", will be of great importance for us in section 4.4, specifically, in showing Lemma 4.4.4. Regarding the last two items above, note that the first relates directly the *stirring process*, $\underline{x}$, and the "times" process $\underline{y}$, where, fixed a particle i in the stirring process, we may only look at all the different particles in $\underline{y}$. The usefulness of the last one is clear: if we manage to choose the priority of a specific particle as the lowest, then we may only look at the "independent" process. These two, together, allow us to, under some conditions, treat the *stirring process* as a system of *independent* particles.

Theorem 4.1.11. *Let $T > 0$ and $\underline{x}(0) = \underline{x}^0(0)$. Then, for any $\xi > 0$ and k there is c such that for all $t \leq T$ and for all $\epsilon > 0$*

$$\mathbf{P}_\epsilon[\left|x_\ell - x^0_\ell(t)\right| \geq (\epsilon^{-2}t)^{1/4+\xi}] \leq c(\epsilon^{-2}t)^{-k}. \tag{4.1.18}$$

4.2 Integral inequalities for the v-functions

In this section, we will start by deriving a particular stochastic equation for the $v-$function. For simplicity, we will denote $v(\underline{x},t) \equiv v^\epsilon(\underline{x}, t \mid \mu)$. We recall from the previous chapter that $N^2\mathcal{L}_{N,0}\eta_{tN^2}(x) = \triangle_N \eta_{tN^2}(x)$, and thus by Kolmogorov's equation (2.0.32), we have for the *stirring process* only $\frac{\partial}{\partial t}\rho^N_t(x) = \triangle_N \rho^N_t(x)$. Here, we will derive a "similar" expression for the v-functions. To take advantage of the sharp gradient estimate in (E.1.4), and to exploit the smallness of the gradients of the v-functions, we will show that $\frac{\partial}{\partial t}v(X,t) = \epsilon^{-2}(\mathfrak{L}_0 v)(X,t) + (C^{(\theta)}_\epsilon v)(X,t)$, where $\mathfrak{L}_0$ is now the $n-$dimensional reflected laplacian. In order to do this, we gain an error $(C^{(\theta)}_\epsilon v)(X,t)$ from both the bulk and boundary dynamics. We "treat" this error by writing it as a function of the gradients of ρ_ϵ and $v(X,t)$ as much as possible. After this, we will apply *Duhamel's formula* , in order to get an integral expression of the aforementioned differential equation. Settled this integral formulation, we will start to bound the $v-$function. In this integral form, we will have transition probabilities, that we will bound with the results of Appendix **E**. Every bound (that is, for the error arising from the boundary dynamics and from the bulk dynamics) will again be a function of both our $v-$functions and ρ_ϵ, but with the difference that the $v-$function will have either *less* or *more* particles. After writting the *labeled* version of these bounds, we show, with an iterative argument, that either we may apply these bounds indefinetly, or at some point we are left with no particles, thus paving the way for the next section.

Definition 4.2.1 (A, B, and C operators)**.** For $X \subset \Lambda_N, t > 0$ we define the linear operator A acting on v such that $(Av)(X,t) = 0$, if $|X| \leq 1$ otherwise, it is equal to

$$(Av)(X,t) = \sum_{\substack{x,x\in X\\ x\sim y}} \left[(\rho_\epsilon(x,t) - \rho_\epsilon(y,t))(v(X\backslash\{x\},t) - v(X\backslash\{y\},t)) - (\rho_\epsilon(x,t) - \rho_\epsilon(y,t))^2 v(X\backslash(x\cup y),t)\right]. \tag{4.2.1}$$

For $b(Z, Z', t) \in \mathbb{R}$ to be specified later on, with $Z, Z' \subset I_+$ or $Z, Z' \subset I_-$ we define B:

$$(B_\pm v)(X,t) := \sum_{Z'\subset I_\pm} b_\pm(X\cap I_\pm, Z', t)v(X\backslash[x\cap I_\pm]\cup Z', t). \tag{4.2.2}$$

Letting $Bv := (B_+ + B_-)v$ we define C as

$$(C_\epsilon^{(\theta)}v)(X,t) := \epsilon^{-2}((Av)(X,t) + \epsilon^\theta(Bv)(X,t)). \tag{4.2.3}$$

For simplicity, let $\overline{\eta}_\epsilon(X,t) := \prod_{x\in X}(\eta(x,t) - \rho_\epsilon(x,t))$. Then, for a set $X \subset \Lambda_N$ we define the v-functions by $v(X,t) = \mathbf{E}_\epsilon \prod_{x\in X}(\eta(x,t) - \rho_\epsilon(x,t)) \equiv \mathbf{E}_\epsilon\overline{\eta}_\epsilon(X,t)$. We note that the v-function is symmetric in the variables $\{x_i : i = 1, \cdots, n\}$. Therefore we consider it defined in the set of points of the form $\{(x_1, \cdots, x_n) : 1 \leq x_1 < x_2 < \cdots < x_n \leq N-1\}$. We extend the definition of the v-function to the boundary of the previous set by stating that it is equal to zero when restricted to it. Therefore, the v-function $v(X,t)$ is defined on sets of the form

$$V_N^n = \{(x_1, \cdots, x_n) \in \{0, \cdots, N\}^n : 1 \leq x_1 < x_2 < \cdots < x_n \leq N-1\}. \tag{4.2.4}$$

Observe that when $n = 2$, the aforementioned set above is simply given by

$$V_N^2 = \{(x,y) \in \{0, \cdots, N\}^2 : 0 < x < y < N\}, \tag{4.2.5}$$

and its boundary $\partial V_N^2 = \{(x,y) \in \{0, \cdots, N\}^2 : x = 0 \text{ or } y = N\}$. In dimension d it is a simplex.

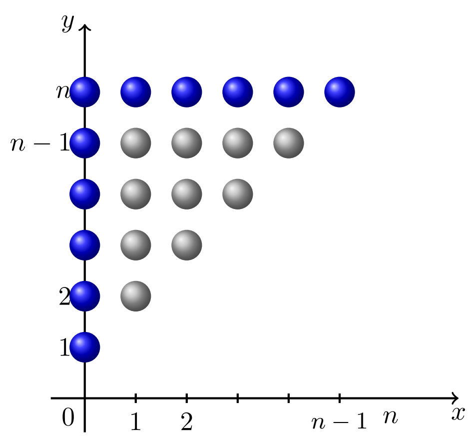


Figure 4.1: The set V_N and its boundary ∂V_N.

4.2.1 A discrete equation for the v-functions

Now we derive the discrete equation for the v-functions.

Lemma 4.2.2.

$$\frac{\partial}{\partial t} v(X,t) = \epsilon^{-2}(\mathfrak{L}_0 v)(X,t) + (C_\epsilon^{(\theta)} v)(X,t) \tag{4.2.6}$$

where the coefficients b *in the definition of* $C_\epsilon^{(\theta)}$ *(specifically, in* $B_\pm$*) are such that* $b(\emptyset, Z', t) = 0$*, if* $|Z| = 1$ *then* $b(Z, \emptyset, t) = 0$*, and*

$$\forall M \in \mathbb{Z} : \sup_{t, |Z| \le M, |Z'| \le M} |b(Z, Z', t)| < \infty \tag{4.2.7}$$

Above the operator $\mathfrak{L}_0$ *is the reflected* d*-dimensional discrete Laplacian defined as follows:*

$$\mathfrak{L}_0 f(x_1, \cdots, x_n) = \sum_{i=1}^{n} \sum_{|y - x_i| = 1} \{f(x_1, \cdots, y, \cdots, x_n) - f(x_1, \cdots, x_i, \cdots, x_n)\}$$

if for all $i \in \{1, \cdots, n\}$ $|x_i - x_{i+1}| > 1$*, otherwise, if there exists* j *such that* $|x_j - x_{j+1}| = 1$*, then*

$$\begin{aligned} \mathfrak{L}_0 f(x_1, \cdots, x_n) = & \sum_{\substack{i=1 \\ i \ne j, j+1}}^{n} \sum_{|y - x_i| = 1} \{f(x_1, \cdots, y, \cdots, x_n) - f(x_1, \cdots, x_i, \cdots, x_n)\} \\ & + \{f(\cdots, x_j - 1, x_j + 1, \cdots) + f((\cdots, x_j, x_j + 2, \cdots) - f(\cdots, x_j, x_j + 1, \cdots)\}. \end{aligned} \tag{4.2.8}$$

When the points $(x_1, \cdots, x_n)$ *are close to the boundary the operator* $\mathfrak{L}_0$ *is simply* N *times the discrete derivative.*

Proof. Let $\overline{\eta}_\epsilon(X,t) := \prod_{x \in X} (\eta(x,t) - \rho_\epsilon(x,t))$, where η is our exclusion process, and ρ_ϵ is the solution of (4.2.6).

Then by Dynkin's formula (2.0.33) we have

$$\frac{d}{dt} v(X,t) = \mathbf{E}_\epsilon \left[\Big(\mathcal{L}_\epsilon + \frac{\partial}{\partial t} \Big) \overline{\eta}_\epsilon(X,t) \right] \tag{4.2.9}$$

where the partial derivative acts on ρ_ϵ and the generator on η. By the product rule we have

$$\mathbf{E}_\epsilon[\frac{\partial}{\partial t} \overline{\eta}_\epsilon(X,t)] \equiv \mathbf{E}_\epsilon \left[\frac{\partial}{\partial t} \prod_{x \in X} (\eta(x,t) - \rho_\epsilon(x,t)) \right] = -\sum_{x \in X} \frac{\partial}{\partial t} \rho_\epsilon(x,t) \mathbf{E}_\epsilon[\overline{\eta}_\epsilon(X \backslash x, t)] \tag{4.2.10}$$

thus,

$$\frac{\partial}{\partial t} v(X,t) = \mathbf{E}_\epsilon[\epsilon^{-2}(\mathcal{L}_0 + \epsilon^\theta \mathcal{L}_\pm) \overline{\eta}_\epsilon(X,t)] - \mathbf{E}_\epsilon[\sum_{x \in X} \overline{\eta}_\epsilon(X \backslash x, t) \frac{\partial}{\partial t} \rho_\epsilon(x,t)]. \tag{4.2.11}$$

Recalling that we have an expression for $\frac{\partial}{\partial t} \rho_\epsilon(x,t)$ from (4.1.3), we claim that we can rearrange the bulk terms to:

$$\epsilon^{-2} \mathbf{E}_\epsilon[\mathcal{L}_0 \overline{\eta}_\epsilon(X,t)] - \mathbf{E}_\epsilon[\sum_{x \in X} \overline{\eta}_\epsilon(X \backslash x, t) \triangle_\epsilon \rho_\epsilon(x,t)] = \epsilon^{-2}(\mathcal{L}_0 v(X,t) + (Av)(X,t)). \tag{4.2.12}$$

Before we proceed we explain how to obtain last display. First let us suppose that X does not contain any pair of neighbour points. Then,

$$\mathcal{L}_0\overline{\eta}_\epsilon(X,t) = \sum_{x\in X} v(X\setminus\{x\},t)\mathcal{L}_0\tilde{\eta}(x,t) = \sum_{x\in X} v(X\setminus\{x\},t)\Delta\eta(x,t). \tag{4.2.13}$$

From last identity we conclude that (4.2.12) is equal to $\mathbf{E}_\epsilon \sum_{x\in X}\overline{\eta}_\epsilon(X\backslash x,t)\Delta_\epsilon\tilde{\eta}(x,t)$,and by the definition of the operator $\mathfrak{L}_0$ last expression is equal to $\epsilon^{-2}(\mathfrak{L}_0 v)(X,t)$.

Now we analyse the case when there are at least two neighbouring points in the set X. To simplify the exposition, let us consider the set $X=\{x,x+1,y\}$, where y is not neighbour of x nor $x+1$. The generalization to other types of sets is completely analogous. Note that

$$\begin{aligned}
\mathcal{L}_0\overline{\eta}_\epsilon(X,t) &= \mathcal{L}_0\Big(\tilde{\eta}(x,t)\tilde{\eta}(x+1,t)\tilde{\eta}(y,t)\Big) = \tilde{\eta}(x,t)\tilde{\eta}(x+1,t)\Delta\eta(y,t)\\
&+\tilde{\eta}(y,t)\mathcal{L}_0\Big(\eta(x,t)\eta(x+1,t)-\rho_\epsilon(x,t)\eta(x+1,t)-\rho_\epsilon(x+1,t)\eta(x,t)\Big)\\
&=\tilde{\eta}(x,t)\tilde{\eta}(x+1,t)\Delta\eta(y,t)\\
&+\tilde{\eta}(y,t)\Big(\eta(x-1,t)\eta(x+1,t)+\eta(x,t)\eta(x+2,t)-2\eta(x,t)\eta(x+1,t)\Big)\\
&-\tilde{\eta}(y,t)\rho_\epsilon(x,t)\Delta\eta(x+1,t)-\tilde{\eta}(y,t)\rho_\epsilon(x+1,t)\Delta\eta(x,t).
\end{aligned}$$

By writing the term on the fifth line of last display in terms of the variables $\tilde{\eta}$ we get

$$\begin{aligned}
\mathcal{L}_0\overline{\eta}_\epsilon(X,t) &= \tilde{\eta}(x,t)\tilde{\eta}(x+1,t)\Delta\eta(y,t)\\
&+\tilde{\eta}(y,t)\Big(\tilde{\eta}(x-1,t)\tilde{\eta}(x+1,t)+\tilde{\eta}(x,t)\tilde{\eta}(x+2,t)-2\tilde{\eta}(x,t)\tilde{\eta}(x+1,t)\Big)\\
&+\tilde{\eta}(y,t)\Big(\rho_\epsilon(x+1,t)\eta(x-1,t)+\rho_\epsilon(x-1,t)\eta(x+1,t)-\rho_\epsilon(x+1,t)\rho_\epsilon(x-1,t)\Big)\\
&+\tilde{\eta}(y,t)\Big(\rho_\epsilon(x,t)\eta(x+2,t)+\rho_\epsilon(x+2,t)\eta(x,t)-\rho_\epsilon(x+2,t)\rho_\epsilon(x,t)\Big)\\
&+\tilde{\eta}(y,t)\Big(\rho_\epsilon(x+1,t)\eta(x,t)+\rho_\epsilon(x,t)\eta(x+1,t)-\rho_\epsilon(x+1,t)\rho_\epsilon(x,t)\Big)\\
&-\tilde{\eta}(y,t)\rho_\epsilon(x,t)\Delta\eta(x+1,t)-\tilde{\eta}(y,t)\rho_\epsilon(x+1,t)\Delta\eta(x,t).
\end{aligned}$$

Note that the expectation with respect to $\mathbf{E}_\epsilon$ of the two terms on the right hand side of the last display is equal to $(\mathfrak{L}_0 v)(X,t)$. Note also that, in this case, the second term at the right hand side of (4.2.12) is equal to

$$\mathbf{E}_\epsilon\tilde{\eta}(x,t)\tilde{\eta}(x+1,t)\triangle_\epsilon\rho_\epsilon(y,t)+\mathbf{E}_\epsilon\tilde{\eta}(y,t)\tilde{\eta}(x+1,t)\triangle_\epsilon\rho_\epsilon(x,t)+\mathbf{E}_\epsilon\tilde{\eta}(y,t)\tilde{\eta}(x,t)\triangle_\epsilon\rho_\epsilon(x+1,t).$$

Putting together the previous computations, by a tedious, but simple computation, we can conclude that (4.2.12) is equal to

$$\begin{aligned}
&\mathbf{E}_\epsilon(\mathfrak{L}_0 v)(X,t)\\
&+\mathbf{E}_\epsilon\tilde{\eta}(y,t)\Big(\rho_\epsilon(x,t)\eta(x+2,t)+\rho_\epsilon(x+2,t)\eta(x,t)-\rho_\epsilon(x+2,t)\rho_\epsilon(x,t)\Big)\\
&+\mathbf{E}_\epsilon\tilde{\eta}(y,t)\Big(\rho_\epsilon(x+1,t)\eta(x,t)+\rho_\epsilon(x,t)\eta(x+1,t)-\rho_\epsilon(x+1,t)\rho_\epsilon(x,t)\Big)
\end{aligned}$$

$$-\mathbf{E}_\epsilon\tilde{\eta}(y,t)\rho_\epsilon(x,t)\Delta\eta(x+1,t)-\mathbf{E}_\epsilon\tilde{\eta}(y,t)\rho_\epsilon(x+1,t)\Delta\eta(x,t)$$

$$-\mathbf{E}_\epsilon\tilde{\eta}(x,t)\tilde{\eta}(x+1,t)\triangle_\epsilon\rho_\epsilon(y,t)+\mathbf{E}_\epsilon\tilde{\eta}(y,t)\tilde{\eta}(x+1,t)\triangle_\epsilon\rho_\epsilon(x,t)+\mathbf{E}_\epsilon\tilde{\eta}(y,t)\tilde{\eta}(x,t)\triangle_\epsilon\rho_\epsilon(x+1,t).$$

A simple computation shows that last terms can be written exactly as defined in the operator A and this proves the claim. Now, we are left with the boundary terms:

$$\epsilon^\theta\left(\mathbf{E}_\epsilon\mathcal{L}_\pm\overline{\eta}_\epsilon(X,t)-\mathbf{E}_\epsilon\sum_{x\in X}\overline{\eta}_\epsilon(X\backslash x,t)D_\pm\rho_\epsilon(x,t)\right). \tag{4.2.14}$$

We will study the term $D_{+,1}\rho_\epsilon(x,t)$ only, since for the others the analysis is completely analogous. Recalling that the generator $\mathcal{L}_+$ acts on each $\eta(x)$ we have

$$\mathbf{E}_\epsilon\mathcal{L}_+\overline{\eta}_\epsilon(X,t)=\mathbf{E}_\epsilon\sum_{x\in X}\overline{\eta}_\epsilon(X\backslash x,t)\mathcal{L}_+\eta(x,t), \tag{4.2.15}$$

where we recall that $\epsilon^\theta\mathcal{L}_+\eta(x,t)=\epsilon^\theta(\beta D_{+,1}\eta(x,t)-\delta D_{+,0}\eta(x,t))$. That is, we will study

$$\epsilon^\theta\left(\beta\mathbf{E}_\epsilon\sum_{x\in X}\overline{\eta}_\epsilon(X\backslash x,t)D_{+,1}\eta(x,t)-\mathbf{E}_\epsilon\sum_{x\in X}\overline{\eta}_\epsilon(X\backslash x,t)D_{+,1}\rho_\epsilon(x,t)\right). \tag{4.2.16}$$

In particular, we are interested in the quantity $\beta D_{+,1}\eta(x,t)-D_{+,1}\rho_\epsilon(x,t)$. In the end we want to have $v-$functions, thus we want to express the aforementioned quantity as a function of ρ_ϵ as much as possible. Thus,

$$D_{+,1}\eta(x,t)=(1-\rho_\epsilon(x,t)-(\eta(x,t)-\rho_\epsilon(x,t)))\prod_{y=x+1}^{N-1}\left((\eta(y,t)-\rho_\epsilon(y,t))+\rho_\epsilon(y,t)\right). \tag{4.2.17}$$

By the distributive property the product term expands to all the combinations of products of $|I_+(x+1)|$ different elements:

$$\prod_{y=x+1}^{N-1}\left((\eta(y,t)-\rho_\epsilon(y,t))+\rho_\epsilon(y,t)\right)=\sum_{Z\in\mathcal{P}(I_+(x+1))}\prod_{z\in Z^c}\rho_\epsilon(z,t)\prod_{z\in Z}(\eta(z,t)-\rho_\epsilon(z,t)), \tag{4.2.18}$$

where $\mathcal{P}(I_+(x+1))$ are the parts of $I_+(x+1)$. Now we separate the case where $Z=\emptyset$, that is, Z^c are the "diagonal" elements:

$$\prod_{z\in I_+(x+1)}\rho_\epsilon(z,t)+\sum_{\substack{Z\in\mathcal{P}(I_+(x+1))\\ Z\neq\emptyset}}\prod_{z\in Z^c}\rho_\epsilon(z,t)\prod_{z\in Z}(\eta(z,t)-\rho_\epsilon(z,t)), \tag{4.2.19}$$

since $\prod_{\emptyset} := 1$. Replacing this in (4.2.16) we get:

$$
\begin{aligned}
&\epsilon^\theta \mathbf{E}_\epsilon \sum_{x\in X} \overline{\eta}_\epsilon(X\backslash x,t)(\beta D_{+,1}\eta(x,t) - D_+\rho_\epsilon(x,t)) = \\
&= \epsilon^\theta \mathbf{E}_\epsilon \sum_{x\in X} \overline{\eta}_\epsilon(X\backslash x,t)(\beta(1-\rho_\epsilon(x,t) - (\eta(x,t)-\rho_\epsilon(x,t)))(\prod_{z\in I_+(x+1)} \rho_\epsilon(z,t) + \\
&+ \sum_{\substack{Z\in\mathcal{P}(I_+(x+1))\\ Z\neq\emptyset}} \prod_{z\in Z^c} \rho_\epsilon(z,t) \prod_{z\in Z}(\eta(z,t)-\rho_\epsilon(z,t))) - D_+\rho_\epsilon(x,t)) \\
&= -\epsilon^\theta\beta \sum_{x\in X} v(X,t) \prod_{z\in I_+(x+1)} \rho_\epsilon(z,t) - \epsilon^\theta\beta \sum_{x\in X} \sum_{\substack{Z\in\mathcal{P}(I_+(x+1))\\ Z\neq\emptyset}} \prod_{z\in Z^c} \rho_\epsilon(z,t) v(Z\cup X,t) - \\
&- \epsilon^\theta(1-\beta)\sum_{x\in X} v(X\backslash x,t) D_+\rho_\epsilon(x,t) + \epsilon^\theta\beta \sum_{x\in X} \sum_{\substack{Z\in\mathcal{P}(I_+(x+1))\\ Z\neq\emptyset}} \prod_{z\in Z^c} \rho_\epsilon(z,t) v(Z\cup[X\backslash x],t).
\end{aligned}
\tag{4.2.20}
$$

At this step, it is not difficult to derive the properties of B. We remark that we only used the term $D_{+,1}$. For $D_{+,0}$, one derives an analagous expression, then after summing both we have the expression in the statement. Doing the computations for the right boundary we are done. □

4.2.2 Integral inequalities for the $v-$functions

By Duhamel's formula and since $v(X,t)$ is solution of (4.2.6) with $v(X,0) = 0$, which is a consequence of the fact that both the stirring process and the process in (4.1.3) start from the same configuration. We know that

$$
v(X,t) = \mathbb{E}_X\Big[\int_0^t (C_\epsilon^{(\theta)}v)(X(s),t-s))\,ds\Big] = \int_0^t S_s(C_\epsilon^{(\theta)}v)(X(s),t-s)ds \tag{4.2.21}
$$

where S_t is the semigroup associated to the process $X(t)$. We recall our discussion regarding the dual process, in (2.0.43), and Proposition 4.1.3. To express the v-function in terms of transition probabilities for the process $X(t)$, we define

$$
P_\epsilon(X \xrightarrow{s} Y) := P_\epsilon(X(s) = Y \mid X(0) = X), \tag{4.2.22}
$$

for $X, Y \subset \Lambda_N$ where $|X| = |Y|$. That is, $P_\epsilon(X \xrightarrow{s} Y)$ is the probability that the process starts in X and arrives in Y at time s. By partitioning the state space for the process at the time s, we can write

$$
v(X,t) = \mathbb{E}_X\Big[\int_0^t \sum_Y (C_\epsilon^{(\theta)}v)(Y,t-s))\mathbf{1}_{X(s)=Y}\,ds\Big].
$$

Using the linearity of the expectation and the fact that $\mathbb{E}_X[\mathbf{1}_{X(s)=Y}] = P_\epsilon(X \xrightarrow{s} Y)$ we, conclude that

$$
v(X,t) = \int_0^t \sum_Y (C_\epsilon^{(\theta)})(Y,t-s)P_\epsilon(X \xrightarrow{s} Y)\,ds. \tag{4.2.23}
$$

On the following lemma we bound the terms associated to $B_\pm$:

Lemma 4.2.3. *Let*

$$\psi_u := \epsilon^{\theta-2} \sum_{Y,Z'\subset I_u} P_\epsilon(X \xrightarrow{s} Y) 1_{\{Y\cap I_u \neq \emptyset\}} b(Y\cap I_u, Z') v([Y\backslash(Y\cap I_u)]\cup Z', t-s). \tag{4.2.24}$$

Then $\forall n, \xi > 0 \quad \exists c : \forall X \subset \Lambda_N, |X| = n, s < t < \log\bigl(\epsilon^{-1}\bigr)$ *we have, for* $u = \pm$

$$|\psi_u| \leq \sum_{\substack{Z'\subset I_u\\ \emptyset\neq X''\subset X\\ W\subset I_u^c}} 1_{\{|Z'|=0,|X''|=1\}^c} \frac{c\epsilon^{\theta-2}}{(\epsilon^{-2}s)^{|X''|/2}+1} P_\epsilon(X\backslash X'' \xrightarrow{s} W)|v(W\cup Z', t-s)|. \tag{4.2.25}$$

Proof. Again, we consider only $B_{+,1}$. Recalling Andjel's inequality, (E.2.1), we *decompose* Y into an union of two elements, one contained in the window I_+ and another contained in the bulk: $Y = W \cup Z$, where $W \subset I_+^c$, $Z \subset I_+, |Z| > 0$ and clearly, $|W \cup Z| = |Y|$. For simplicity, let us denote $\eta(X) := \prod_{x\in X} \eta(x)$. In this way the transition probabilities decompose to

$$\begin{aligned} P_\epsilon(X \xrightarrow{s} Y) = P(X \xrightarrow{s} W\cup Z) &\equiv P_\epsilon(X(s) = Z\cup W \mid X(0) = X) \\ &= P_\epsilon(\eta_s(Z\cup W) \mid \eta_0(X)). \end{aligned} \tag{4.2.26}$$

Then, by (E.2.1) we get

$$P_\epsilon(X \xrightarrow{s} Y) = P(\eta_s(Z) = 1 \mid \eta_0(X) = 1) P(\eta_s(W) = 1 \mid \eta_0(X) = 1). \tag{4.2.27}$$

Now note that

$$\begin{aligned} &P(\eta_s(Z) = 1 \mid \eta_0(X) = 1) = \\ &= \sum_{\substack{Y':\\ |Y'|=|X|\\ Z\subset Y'}} P(\eta_s(Z) = 1 \mid \eta_s(Y') = 1, \eta_0(X) = 1) P(\eta_s(Y') = 1 \mid \eta_0(X) = 1) \leq \sum_{Y'\supset Z} P_\epsilon(X \xrightarrow{s} Y'), \end{aligned} \tag{4.2.28}$$

while for W,

$$\begin{aligned} &P(\eta_s(W) = 1 \mid \eta_0(X) = 1) \\ &= \sum_{\substack{Y:\\ |Y|=|X|\\ W\subset Y}} P(\eta_s(W) = 1 \mid \eta_s(Y) = 1, \eta_0(X) = 1) P(\eta_s(Y) = 1 \mid \eta_0(X) = 1) \\ &\leq \sum_{Y\supset W} P_\epsilon(X \xrightarrow{s} Y) \end{aligned} \tag{4.2.29}$$

and we can see that

$$P_\epsilon(X \xrightarrow{s} Y) \leq \left(\sum_{Y'\supset Z} P_\epsilon(X \xrightarrow{s} Y')\right)\left(\sum_{Y\supset W} P_\epsilon(X \xrightarrow{s} Y)\right), \tag{4.2.30}$$

by applying Andjel's inequality to each W and Z fixed, then bounding each probability by their sum. By *Ligget's* *inequality* ([21], pages $226-228$), we have

$$P_\epsilon(X \xrightarrow{s} Y') \le \prod_{x\in X}\sum_{y\in Y'} P(x \xrightarrow{s} y). \tag{4.2.31}$$

From (E.1.2) we have that for $X=\{x\}, Y=\{y\}$ we have $P_\epsilon(X \xrightarrow{s} Y') = P_s^{(\epsilon)}(x,y)$. Thus, we want to bound

$$\sum_{Y'\supset Z}\prod_{x\in X}\sum_{y\in Y'} P_s^{(\epsilon)}(x,y). \tag{4.2.32}$$

By (E.1.2) this is bounded by $\frac{c}{(\epsilon^{-2}s)^{-|Z|/2+1}}$. To see this, first note that (in (E.1.2)) $x-y\ge 1$. Doing a taylor expansion in the exponential function and multiplying the terms, and noticing that $(\epsilon^{-2}+1)^{|Z|}\ge \epsilon^{|Z|/2}+1$ the bound follows. Now rewrite the W term to the transition probability into W:

$$\sum_{Y\supset W} P_\epsilon(X \xrightarrow{s} Y) = \sum_{X'\subset X} P_\epsilon(X' \xrightarrow{s} W) \tag{4.2.33}$$

and we can now bound $|\psi_+|$ as follows:

$$\begin{aligned}
&\epsilon^{\theta-2}\sum_{\substack{Y,Z'\subset I_+\\ Y\cap I_+\neq\emptyset}} P_\epsilon(X \xrightarrow{s} Y)|b(Y\cap I_+,Z')||v([Y\backslash(Y\cap I_+)]\cup Z',t-s)|\\
&\le \sum_{\substack{Y,Z'\subset I_+\\ Y\cap I_+\neq\emptyset}} \frac{c\epsilon^{\theta-2}}{(\epsilon^{-2}s)^{|Z|/2}+1}\Big(\sum_{\substack{X'\subset X\\ |X'|=|W|}} P_\epsilon(X' \xrightarrow{s} W)\Big)|b(Y\cap I_+,Z')||v([Y\backslash(Y\cap I_+)]\cup Z',t-s)|.
\end{aligned} \tag{4.2.34}$$

Noticing that $Y\cap I_+ = Z$, $Y\backslash[Y\cap I_+]\cap Z' = W$ and recalling that $|b(Z,Z')|\le c1_{\{|Z|=1,|Z'|=0\}^c}$ we can bound the sum in Z' of the b terms by $\mathcal{P}(K)$. By the decomposition of Y, summing in Y is the same as summing in W and Z, thus

$$\sum_{Z'\subset I_+}\sum_{\emptyset\neq Z\subset I_+}\sum_{\substack{W\subset I_+^c:\\ |W\cup Z|=|X|}} \frac{c'\epsilon^{\theta-2}\mathcal{P}(K)}{(\epsilon^{-2}s)^{|Z|/2}+1}\sum_{\substack{X'\subset X\\ |X'|=|W|}} P_\epsilon(X' \xrightarrow{s} W)1_{\{|Z|=1,|Z'|=0\}^c}|v(W\cup Z',t-s)|. \tag{4.2.35}$$

Letting $X' = X\backslash X''$: $|W| = |X\backslash X''|$, we have the bound in the statement. □

We remark that the contribution of $\epsilon^{-2}(Av)(X,t)$ still needs to be bounded. Before that, we will write the labeled version of the bound in the previous lemma, and of $(Av)(X,t)$. For that, we order *arbitrarily* the sites of X, which will be denoted by $\underline{x} = (x_1\dots,x_n)$. As seen before, $v(X,t)$ is symmetric under the exchange of position of particles, thus setting $v(\underline{x},t) := v(X,t)$, we have that $v(\underline{x},t)$ is symmetric under exchange of labels. We denote by $\mathbb{E}_{\epsilon,\underline{x}}$ the expectation with respect to the *stirring process* starting at time 0 from $\underline{x}$ (*i.e.,* $\underline{x}\equiv\underline{x}(0)$). We will sometimes write $\mathbf{E}_{\epsilon,\underline{x}}\equiv\mathbf{E}_\epsilon$ instead, when it is clear from the context. In this way, we denote the labeled version of the following sets as $X''\equiv J$ and $Z'\equiv\underline{z}'$, and noticing that the labeled version of

$$\sum_{\substack{W\subset I_+^c:\\ |W|=|X\backslash X''|}} P_\epsilon(X\backslash X'' \xrightarrow{s} W)|v(W\cup Z',t-s)| \quad\text{is}\quad \mathbf{E}_{\epsilon,\underline{x}}\left[1_{\{\underline{x}^{(J)}(s)\subset I_\pm^c\}}\left|v(\underline{x}^{(J)}(s)\cup\underline{z}',t-s\right|\right], \tag{4.2.36}$$

we can rewrite the bound in Lemma 4.2.3 as:

$$\sum_{\emptyset\neq J\subset\{1,\dots,n\}}\sum_{\underline{z}'\subset I_\pm} 1_{\{|J|=1,|z'|=0\}^c}\frac{c\epsilon^{\theta-2}}{(\epsilon^{-2}s)^{|J|/2}+1}\mathbf{E}_{\epsilon\underline{x}}\left[1_{\{\underline{x}^{(J)}(s)\subset I_\pm^c\}}\left|v(\underline{x}^{(J)}(s)\cup\underline{z}',t-s)\right|\right], \tag{4.2.37}$$

where $\underline{x}^{(J)}$ is the labeled set X except the particles in J. Moreover, the labeled version of (Av) takes the form:

$$(Av)(\underline{x},t) := \sum_{\substack{x_i,x_j\in\underline{x}\\ x_i\sim x_j}}\left[(\rho_\epsilon(x_i,t)-\rho_\epsilon(x_j,t))(v(\underline{x}^{(i)},t)-v(\underline{x}^{(j)},t))-(\rho_\epsilon(x_i,t)-\rho_\epsilon(x_j,t))^2 v(\underline{x}^{(i,j)},t)\right]. \tag{4.2.38}$$

Remark 4.2.4. Note that on this expression we are evaluating ρ_ϵ and v at time t with respect to the labeled configuration/set $\underline{x}\equiv\underline{x}(0)$ evolved up to t.

Putting the bound (4.2.37) in (4.2.21) with the labeled (Av) we get:

$$\begin{aligned}|v(\underline{x},t)| \leq \int_0^t ds \Bigg(&\sum_{u=\pm}\sum_{\emptyset\neq J\subset\{1,\dots,n\}}\sum_{\underline{z}'\subset I_u}\frac{c\epsilon^{\theta-2}}{(\epsilon^{-2}s)^{|J|/2}+1}1_{\{|J|=1,|z'|=0\}^c}\times\\ &\times\mathbf{E}_\epsilon\left[1_{\{\underline{x}^{(J)}(s)\subset I_u^c\}}\left|v(x^{(J)}(s)\cup\underline{z}',t-s\right|\right]+\epsilon^{-2}\mathbf{E}_\epsilon(Av)(\underline{x}(s),t-s)\Bigg).\end{aligned} \tag{4.2.39}$$

Remark 4.2.5. Note that it is the same as erasing the particles $x_j, j\in J$ either at time 0 or s. The particles can be erased in the beggining since their labels must be *completely new*.

Recalling (E.1.4), we bound the gradients of ρ_ϵ in $(Av)(\underline{x},t-s)$ as in (4.4.10), and the squares of the gradients as

$$\left(\frac{c'}{(\epsilon^{-2}t)^{1/2-\xi'}+1}\right)^2 \leq \frac{c}{(\epsilon^{-2}t)^{1-\xi}+1}, \tag{4.2.40}$$

by noticing that

$$((\epsilon^{-2}t)^{1/2-\xi'}+1)^2 = (\epsilon^{-2}t)^{1-2\xi'}+1+2(\epsilon^{-2}t)^{1/2-\xi'}+1 \geq (\epsilon^{-2}t)^{1-\xi}+1. \tag{4.2.41}$$

In this way we have the following bound for (Av):

$$\mathbf{E}_\epsilon|(Av)(\underline{x}(s),t-s)| \leq c\sum_{\substack{x_i,x_j\in\underline{x}\\ x_i\sim x_j}}\mathbf{E}_\epsilon\left[\frac{v(\underline{x}^{(i)}(s),t-s)-v(\underline{x}^{(j)}(s),t-s)}{(\epsilon^{-2}(t-s))^{1/2-\xi}+1}+\frac{v(\underline{x}^{(i,j)}(s),t-s)}{(\epsilon^{-2}(t-s))^{1-\xi}+1}\right]. \tag{4.2.42}$$

Now we proceed to bound the "gradients" $v(\underline{x}^{(i)},t)-v(\underline{x}^{(j)},t)$. For that, we will use the *A/P process* and, more specifically, Lemma 4.1.6, by noticing that the function

$$f_{i,j}(\underline{x},s) := (C_\epsilon^{(\theta)}v)(\underline{x}(s)\backslash x_i(s),t-s)-(C_\epsilon^{(\theta)}v)(\underline{x}(s)\backslash x_j(s),t-s) \tag{4.2.43}$$

is antisymmetric under the exchange of i,j.

Lemma 4.2.6.

$$\left|v(\underline{x}^{(i)},t)-v(\underline{x}^{(j)},t)\right| \le \int_0^t ds\mathbf{E}_\epsilon\left[1_{\{\tau_{i,j}\ge s/2\}}\times\right.$$
$$\left.\times\sum_{\underline{y}}\left\{P_\epsilon(\underline{x}^{(j)}(s/2)\xrightarrow{s/2}\underline{y})+P_\epsilon(\underline{x}^{(i)}(s/2)\xrightarrow{s/2}\underline{y})\right\}\left|C_\epsilon^{(\theta)}(\underline{y},t-s)\right|\right]. \tag{4.2.44}$$

Remark 4.2.7. $P_\epsilon(\underline{x}^{(j)}(s/2)\xrightarrow{s/2}\underline{y}) = P_\epsilon(\underline{x}(s/2)=\underline{y}\mid\underline{x}(0)=\underline{x}^{(j)}(s/2))$ where $\underline{x}^{(j)}(s/2)$ is the labeled process starting from some $\underline{x}'(0)$ with particle j removed. So this means that we started from $\underline{x}'$, evolved it up to $s/2$, then set $\underline{x}=\underline{x}^{(j)}(s/2)$ and evolved it up to $s/2$ again. Thus, $P_\epsilon(\underline{x}^{(j)}(s/2)\xrightarrow{s/2}\underline{y})$ is the probability to arrive at $\underline{y}$ of the latter.

Proof. Recalling Lemma 4.1.6, letting $f_{i,j}(\underline{x},s):=(C_\epsilon^{(\theta)}v)(\underline{x}(s)\backslash x_i(s),t-s)-(C_\epsilon^{(\theta)}v)(\underline{x}(s)\backslash x_j(s),t-s)$, as already mentioned, we have that $f_{i,j}(\underline{x},s)=-f_{j,i}(\underline{x},s)$. Thus, if $\underline{x}(s)$ starts from $\underline{x}$ with both i,j particles we have

$$v(\underline{x}^{(i)},t)-v(\underline{x}^{(j)},t)=\int_0^t ds\mathbf{E}_\epsilon\left[(C_\epsilon^{(\theta)}v)(\underline{x}(s)\backslash x_i(s),t-s)-(C_\epsilon^{(\theta)}v)(\underline{x}(s)\backslash x_j(s),t-s)\right]. \tag{4.2.45}$$

Note that we are removing particle i,j at time s, since our process starts with both of them. By Lemma 4.1.6 we have

$$\left|v(\underline{x}^{(i)},t)-v(\underline{x}^{(j)},t)\right|=\left|\int_0^t ds\mathbf{E}_\epsilon\left[f_{i,j}(\underline{x},s)1_{\{\tau_{i\sim j}>s/2\}}\right]\right| \tag{4.2.46}$$

and we are almost done. Note that we could choose s/a for any $a>1$. The choice $s/2$ is simply to uniformize the bounds that we will get. Since

$$\mathbf{E}_\epsilon\left[1_{\{\tau_{i\sim j}>s/2\}}(C_\epsilon^{(\theta)}v)(\underline{x}(s)\backslash x_i(s),t-s)\right]=$$
$$=\mathbf{E}_\epsilon\mathbf{E}_{\epsilon,\underline{x}^{(i)}(s/2)}\left[1_{\{\tau_{i\sim j}>s/2\}}(C_\epsilon^{(\theta)}v)(\underline{x}(s)\backslash x_i(s),t-s)\right] \tag{4.2.47}$$

we are done. Doing the same for $(C_\epsilon^{(\theta)}v)(\underline{x}(s)\backslash x_j(s),t-s)$ and applying the triangular inequality we have the desired bound. □

Recalling that $C_\epsilon^{(\theta)}:=\epsilon^{-2}(A+\epsilon^\theta B)$, we already have a bound for $\sum_Y P_\epsilon(X\xrightarrow{s}Y)|(B_\pm v)(Y,t-s)|$ from (4.2.37). Applying this bound for $P_\epsilon(\underline{x}^{(i)}(s/2)\xrightarrow{s/2}\underline{y})$ and $P_\epsilon(\underline{x}^{(j)}(s/2)\xrightarrow{s/2}\underline{y})$ we arrive at

$$\left|v(\underline{x}^{(i)},t)-v(\underline{x}^{(j)},t)\right|\le$$
$$\le\int_0^t ds\left(\sum_{u=\pm}\sum_{\emptyset\ne J\subset\{1,\dots,n\}}\sum_{\underline{z}'\subset I_\pm}1_{\{|J|=1,|z'|=0\}^c}\frac{c}{(\epsilon^{-2}s/2)^{|J|/2}+1}\times\right.$$
$$\times\mathbf{E}_\epsilon\left[1_{\{\underline{x}^{(J)}(s)\subset I_\pm^c\}}1_{\{\tau_{i,j}>s/2\}}\left|v(\underline{x}^{(J)}(s)\cup\underline{z}',t-s)\right|\right]+$$
$$\left.+\mathbf{E}_\epsilon\left[1_{\{\tau_{i,j}>s/2\}}\epsilon^{-2}\left\{\left|(Av)(\underline{x}^{(i)}(s),t-s)\right|+\left|(Av)(\underline{x}^{(j)}(s),t-s)\right|\right\}\right]\right), \tag{4.2.48}$$

and we can use (4.2.38) to get a function of v-functions and bound ρ_ϵ as in (E.1.4) again. Now, recalling that we have a bound for A from (4.2.42):

$$\epsilon^{-2}\mathbf{E}_\epsilon(Av)(\underline{x}(s),t-s)\leq$$
$$\leq c\epsilon^{-2}\sum_{\substack{x_i,x_j\in\underline{x}\\ x_i\sim x_j}}\mathbf{E}_\epsilon\left[\frac{v(\underline{x}^{(i)}(s),t-s)-v(\underline{x}^{(j)}(s),t-s)}{(\epsilon^{-2}(t-s))^{1/2-\xi}+1}+\frac{v(\underline{x}^{(i,j)}(s),t-s)}{(\epsilon^{-2}(t-s))^{1-\xi}+1}\right], \tag{4.2.49}$$

we can bound again the $v-$functions as in (4.2.39) by

$$\begin{aligned}|v(\underline{x},t)|\leq\int_0^t ds&\left(\sum_{u=\pm}\sum_{\emptyset\neq J\subset\{1,\dots,n\}}\sum_{\underline{z}'\subset I_u}\frac{c\epsilon^{\theta-2}}{(\epsilon^{-2}s)^{|J|/2}+1}1_{\{|J|=1,|z'|=0\}^c}\times\right.\\ &\times\mathbf{E}_\epsilon\left[1_{\{\underline{x}^{(J)}(s)\subset I_u^c\}}\left|v(x^{(J)}(s)\cup\underline{z}',t-s\right|\right]+\\ &\left.+c\epsilon^{-2}\sum_{\substack{x_i,x_j\in\underline{x}\\ x_i\sim x_j}}\mathbf{E}_\epsilon\left[\frac{v(\underline{x}^{(i)}(s),t-s)-v(\underline{x}^{(j)}(s),t-s)}{(\epsilon^{-2}(t-s))^{1/2-\xi}+1}+\frac{v(\underline{x}^{(i,j)}(s),t-s)}{(\epsilon^{-2}(t-s))^{1-\xi}+1}\right]\right).\end{aligned} \tag{4.2.50}$$

The idea is to iterate the bound for v with the inequalites above, truncating this recurrence at some step m. For a better exposition, let us refer to the (first) bound for $|v(\underline{x},t)|$ as

$$|v_n(\underline{x},t)|\leq\int_0^t dsf_1(v_{n-1},v_{n-2},v_{n-3},v_{n-J+z'},t-s), \tag{4.2.51}$$

where we write v_n since we start with n particles, *i.e.* $|\underline{x}|=n$. Similarly, we control the number of particles with the other indexes on v. By the bound for $\left|v(\underline{x}^{(i)},t)-v(\underline{x}^{(j)},t)\right|$ and replacing the $A's$ by bounds on ρ_ϵ and v, then bounding v and so on, it is easy to see that we have successively bounds of this form:

$$\begin{aligned}&|v_n(\underline{x},t)|\leq\int_0^t dsf_1(v_{n-1},v_{n-2},v_{n-3},v_{n-J_1+z_1'},t-s)\\ &\leq\int_0^t ds\int_{t_1}^t dt_2 f_1(v_{n-2},v_{n-J_1+z_1'},t-s)f_2(v_{n-2},v_{n-3},v_{n-4},v_{n-1-J_2+z_2'},t_2)\\ &\leq\int_0^t ds\int_{t_1}^t dt_2\int_{t_2}^t dt_3 f_1(v_{n-J_1+z_1'},t-s)f_2(v_{n-3},v_{n-4},v_{n-1-J_2+z_2'},t_2)\times\\ &\times f_3(v_{n-3},v_{n-4},v_{n-5},v_{n-2-J_3+z_3'},t_3)\\ &\leq\cdots\end{aligned} \tag{4.2.52}$$

This recursion will be better quantified in the folowing section. Nevertheless, we can already note that we may have some J and z' sets such that our v function is empty in a finite number of iterations. Otherwise, we might iterate the recursion above indefinitely. In this way, it will be important to consider a smart number of iterates and check wheter we already have a nice bound for v, or if eventually our series of bounds explode.

4.3 The Truncated Hierarchy

In order to iterate (4.2.52), we will classify each term arising in the bounds that we derived in the previous section, *i.e.,* starting from the bound for $|v(\underline{x},t)|$, (4.2.50), which is a function of the gradient $\left|v(\underline{x}^{(i)},t)-v(\underline{x}^{(j)},t)\right|$, (4.2.48), and again of $|v|$ but with less particles, *etc.* Thus, we need to classify the coefficients arising when we "remove" 1 particle; remove 2 particles; remove or add a set of particles−and specify these sets. Fixed a number of iterations m, each realization of these sucessive bounds is denoted by *skeleton*, that we define below inductively. Thus, we are interested in the bounds comming from the sum of all these skeletons. In summary, we will associate to each term of each sum of each iteration an index, then we sum over all possible combinations of indexes.

Finally, the series obtained by these finite number of iterations is denoted by *the truncated hierarchy*. Along the way, we will need to quantify the time differences $t - t_i$ and $t_i - t_{i-1}$. This quantification will be artificial and chosen as the more useful and simple possible. Defined this, we will specify *when* each particle is added/removed, through a *branching process*, which in turn is defined by the *A/P-process*. Given the very many sets and sums in each bound, it is good to interpret the skeletons as a *stochastic process* that determines additions and removals of "particles" into/from our system, coupled with the jumps determined by the stirring process and boundary dynamics.

Definition 4.3.1 (The skeleton)**.** Each *skeleton* π is a sequence $\pi = (\pi_i)_{i=1:m(\pi)}$, where $m(\pi) \equiv m \leq M$, for some M to be specified. Each i is a branching "time" (read time being discrete) and, fixed i, π_i denotes which particles die or are born. We will start with the particles $A_0 = \{1, \dots, n\}$ alive. In this way, at each time i we will denote the set of alive particles by A_i, which are determined by the previous values of the skeleton, $\pi_{j \leq i}$ (this notation will be recurrent and denotes the *set* $\{\pi_j\}_{j \leq i}$). For each i, the term π_i is a quadruplet

$$\pi_i = (\delta_i, J_i, u_i, \underline{z}_i) \tag{4.3.1}$$

where

- $\delta_i \in \{0, 1, 2\}$ - determines if we are going to have *births* and/or *deaths* and of which *type*;
- J_i - an increasing sequence of *distinct* integers such that $|J_i| < \infty$ - determines the *set* of particles that *die*;
- $u_i \in \{-, 0, +\}$ - determines *where* particles are born/die;
- $\underline{z}_i$ is a *labeled configuration*, with labels in J_i^+ - the *new* set of particles *i.e., labeled births*.

Say we already know $\pi_{j<i}$. We define inductively π_i as follows:

- $\delta_i = 0, 1 \longrightarrow u_i = 0, \quad \underline{z}_i = \emptyset, \quad J_i = \{k_i, l_i\}$ with $k_i < l_i$,

  $\delta_i = 1 \longrightarrow A_i = A_{i-1} \backslash \{k_i, l_i\}$,

  $\delta_i = 0 \longrightarrow A_i = A_{i-1} \backslash \{l_i\}$ *i.e.,* the particle with the *highest label* in J_i dies.

  Remark 4.3.2. Particles k_i, l_i may *not* be neighbors.

- $\delta_i = 2 \longrightarrow \quad u_i \neq 0, \quad J_i \neq \emptyset \quad A_i = (A_{i-1} \backslash J_i) \cup J_i^+$. Moreover, $|J_i| = 1 \longrightarrow \left|J_i^+\right| > 0$
- $m(\pi) < M \longrightarrow \delta_m > 0, \quad \underline{z}_m = \emptyset, \quad |J_m| = |A_{m-1}|$, that is, $A_m = \emptyset$, and we are considering the case where *all* particles die ($v \equiv v_0$).
- $m(\pi) = M$ then A_M is free: $A_M = \emptyset \vee A_M \neq \emptyset$. It doesn't matter since we will truncate it at this step.

Remark 4.3.3. Note that we are considering $m(\pi)$ to be the *maximum* number of iterations. That is, fixed an integer M large "enough", at $m(\pi)$ we are either without any particle ($m(\pi) < M$), or we truncate our series at the step M, ($m(\pi) = M$). Thus, if $m(\pi) = M$ we may have surviving particles at t_M. Note also that if $A_m = \emptyset$, (*i.e.*, $|\underline{x}(t_m)| = 0$) then we cannot define $\delta_{m+1} = 2$ since we would have $J_{m+1} \neq \emptyset$, neither $\delta_{m+1} = 0, 1$, since we would have $|J_{m+1}| = 2$. Thus, the case where all particles die is well defined.

Now we define the times with respect to the *stirring process* that the particles are removed, according to the skeleton π.

Definition 4.3.4 (The branching process)**.** Given an initial configuration $\underline{x} \equiv \underline{x}(0)$, a path $\omega \in \Omega$ on the *A/P-marks* space, that is, a realization of the process that defines $\underline{x}(\cdot)$, steps $m \equiv m(\pi) \leq M$ and times $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} =: t$, and a skeleton π, we define $\underline{x}(t)$ by following the A/P-process in (t_i, t_{i+1}). At the endpoints, we define as follows. At time t_i:

- $\delta_i = 1 \longrightarrow$ particles $x_j(t_i^-)$ with $j \in J_i$ disappear from $\underline{x}(t_i^-)$;
- $\delta_i = 0 \longrightarrow$ particle with label k_i remains alive, but the one with label l_i dies at the mid point of the time interval $[t,t_{i+1}]$, that is, at time $t_i + (t_{i+1} - t_i)/2$;
- $\delta_i = 2 \longrightarrow$ we require that $x_j(t_i^-) \in I_{u_i}^c$ for all $j \in A_{i-1} \backslash J_i$. At time t_i^+ we then add the particles $\underline{z}_i$.

Remark 4.3.5. Recall that the *A/P-marks* process does not remove particles at the boundaries, with jumps to outside of Λ_N being supressed.

Now that all the terms of the sums are defined in terms of skeletons, and the times at which we remove the particles on the skeletons are defined in terms of the branching process, we will look at which coefficients we will sum at each iteration. For simplicity, $p_i \equiv |J_i|$, and the *events* R_i and T_i are defined as:

$$R_i := \{x_j(t_i^-) \in I_{u_i}^c, j \notin J_i\} \quad \text{and} \quad T_i := \{\tau_{k_i,l_i}(t_i) > (t_{i+1} + t_i)/2\}. \tag{4.3.2}$$

- If $\delta_i = 0$, we associate $1_{\{x_{k_i} \sim x_{l_i}\}} 1_{T_i} \frac{\epsilon^{-2}}{[\epsilon^{-2}(t-t_i)]^{1/2-\xi}+1}$. That is, we are removing *one* particle (l_i), and looking at the (Av) term in the last line of (4.2.48), after the bound from (4.2.49);
- If $\delta_i = 1$, we associate $1_{\{x_{k_i} \sim x_{l_i}\}} \frac{\epsilon^{-2}}{[\epsilon^{-2}(t-t_i)]^{1-\xi}+1}$. That is, we are removing *two* particles ($\{k_i, l_i\}$), and looking at the last term in the last line of (4.2.50).
- If $\delta_i = 2$, we associate $1_{R_i} \frac{\epsilon^{\theta-2}}{[\epsilon^{-2}(t-t_{i-1})]^{p_i/2}+1}$. That is, $\left|J_i^+\right|$ particles are born and $|J_i|$ particles die, as in the second line of (4.2.50).

In this way, bounding the $v-$functions by a constant on the iterations $1 : m = m(\pi)$, we define

$$\begin{aligned} w_\pi(\underline{x}, t) &:= \int_0^t dt_1 \cdots \int_{t_{m-1}}^t dt_m \prod_{|\delta_i=0|} \frac{\epsilon^{-2}}{[\epsilon^{-2}(t-t_i)]^{1/2-\xi}+1} \\ &\prod_{|\delta_i=1|} \frac{\epsilon^{-2}}{[\epsilon^{-2}(t_i - t_{i-1})]^{1-\xi}+1} \prod_{|\delta_i=2|} \frac{\epsilon^{\theta-2}}{[\epsilon^{-2}(t-t_i)]^{p_i/2}+1} \\ &\mathbf{E}_\epsilon \left[\prod_{|\delta_i=0,1|} 1_{\{x_{k_i}(t_i) \sim x_{l_i}(t_i)\}} \prod_{|\delta_i=0|} T_i \prod_{|\delta_i=1|} R_i \right], \end{aligned} \tag{4.3.3}$$

where the products over the sets $|\delta_i = j|$ are defined as $\{i \le m : \delta_i = j\}$. Recalling (4.2.52), it is clear that we have $|v(\underline{x}, t)| \le c \sum_\pi w_\pi(\underline{x}, t)$,for some constant c to be determined, dependent on the "in-out" rates, n and β^*. Of course, for $m(\pi) = M$ we may have $\left|\underline{x}^b(t_M^+)\right| > 0$. These terms do not appear in the definition above because we bounded them by 1.

Observing carefully w_π, note that if we ignore all constants and drop the $+1$ in the denominator, we are left with successive integrals of the following form

$$\int_u^v ds \frac{1}{(s-u)^\alpha (v-s)^\beta}, \tag{4.3.4}$$

with $u < v, \alpha, \beta < 1$, which we know that are finite. Specifically, the integral above can be shown to be equal to $c_{\alpha,\beta}(v-u)^{1-(\alpha+\beta)}$. Thus, we want to control the difference $\triangle t_i := t_{i+1} - t_i$ suitabily. For that, let us take a quantity $\triangle$. Then, $\triangle \ge t - t_m$ or $\triangle > t - t_m$. We say that if $\triangle t_m \le \triangle$, then *the times cluster to* t. Otherwise ($\triangle t_m > \triangle$), we say that *the times do not cluster to* t. As we will see, when the times do not cluster, proofs are simpler. When the times "cluster" to t, we will need to look only at the *last cluster*. For future reference, we state this in the following definition.

Definition 4.3.6 (Cluster to t)**.** Recall that the *truncated hierarchy* induces the partition $[0, t] = \bigcup_{0 \le i \le m} [t_i, t_{i+1}]$,where $t_0 := 0, t_{m+1} := t$, and that $\triangle t_i := t_{i+1} - t_i$. Fixed m, for every skeleton π_m let $\triangle^{\ge} := \{t_i : \quad \triangle \ge \triangle t_i, \quad i \le m\}$.

For such skeleton, for the smaller $0 \le H \le m$ possible, define

$$\mathcal{T}_H := \{t_1, \dots, t_m \mid \forall i \ge H \quad t_i \in \triangle^{\ge}, t_{H-1} \notin \triangle^{\ge}\}. \tag{4.3.5}$$

Then the times $C_H := \{t_H, \dots, t_m, t\}$ are called the *last cluster to* t.
If we have $H = 0$ then $C_H = \emptyset$ and, in particular, we have $\triangle < t - t_m$. In this case, we say that the times do *not* cluster to t.

In this way, denote the integrand in (4.3.3) by $f_{1,\dots,m}$. Then differentiate the quantity in w_π (4.3.3) between when we do not have clusters to t, or when we do, *i.e.*, $w_\pi(\underline{x}, t) = w'_\pi(\underline{x}, t) + w''_\pi(\underline{x}, t)$,where

$$w'_\pi(\underline{x}, t) := \int_0^t dt_1 \cdots \int_{t_{m-1}}^t dt_m 1_{\{t_m < t - \triangle\}} f_{1,\dots,m} \tag{4.3.6}$$

$$w''_\pi(\underline{x}, t) := \int_0^t dt_1 \cdots \int_{t_{m-1}}^t dt_m 1_{\{t_m \ge t - \triangle\}} f_{1,\dots,m}. \tag{4.3.7}$$

Moreover, we can decompose w''_π in quantities associated to each possible *last cluster*, by letting $w_\pi(\underline{x}, t) = \sum_{H \le m} w''_{\pi,H}(\underline{x}, t)$, where the integrand in w''_π is $1_{\mathcal{T}_H} f_{1,\dots,m}$. Note that the skeleton induces a *partition* of $[0, t]$. We will *fix* this partition as the *uniform*, *i.e.*, $t_i - t_{i-1} = \frac{t}{M+1}$. The reason for this is mostly that this partition is simple enough. As we will see in the following section, we will need that $\triangle \le 1$. For that, we will fix

$$\triangle = \frac{t}{M+1} \wedge \epsilon^a, \tag{4.3.8}$$

for some $a > 0$. The choice of a is only needed when we finally bound $w_\pi(\underline{x}, t)$, and therefore, the $v-$function.

Specifically, in (4.5.44). Up to that point, we only need that $\triangle \leq 1$. Nevertheless, let us already fix $a = \frac{K}{K+1}$.

4.4 Bounds for the *skeleton*

Observing (4.3.3), note that we can bound the products in such a way that we get "full powers" of ϵ and $\triangle$ now that we can control the differences $t - t_i$. In this way, the main problem is the expectation terms. We will solve this by bounding inside the expectation, while taking advantage of the bounds that we already have for the events inside the indicator function, from Theorem 4.1.7 and Proposition 4.1.8.

4.4.1 Bounds when times do not cluster

As mentioned in the previous section, we will bound the terms in $w'_\pi(\underline{x}, t)$ in such a way to arrive at the integral (4.3.4). This will be used in the main result of this section:

Proposition 4.4.1. $\forall \xi > 0 \quad \exists c$ *such that* $\forall \pi : m = m(\pi) \leq M, \underline{x} : |\underline{x}| = n, \epsilon > 0, t \leq \epsilon^{\beta^*}$:

$$w'_\pi(\underline{x}, t) \leq c(\epsilon^2 t)^{-\xi M} \triangle^{-S_1(m)} \epsilon^{S_2(m)} t^{S_3(m)} \epsilon^{-S_4^{(\theta)}(m)}, \tag{4.4.1}$$

where, for all $i \leq m$

$$\begin{aligned} S_1(m) &= |\delta_i = 1| + \frac{1}{2}|\delta_i = 0|, \\ S_2(m) &= |\delta_i = 0, 1| + |\delta_i = 2, p_i \geq 2, \delta_{i-1} \neq 0|, \\ S_3(m) &= \frac{1}{2}|\delta_i = 1| + |\delta_i = 2, p_i = 1| + |\delta_i = 2, p_i \geq 2, \delta_{i-1} = 0|, \\ S_4^{(\theta)}(m) &= (1-\theta)\left(|\delta_i = 2, \delta_{i-1} > 0, p_i \geq 2| + |\delta_i = 2, \delta_{i-1} = 0, p_i \geq 2| + |\delta_i = 2, p_i = 1|\right). \end{aligned} \tag{4.4.2}$$

To show this, the main problem is the expectation in the definition of w_π (the last line of (4.3.3)). Thus, we will first show the following auxiliary result. Note that the definitions below are independent of θ.

Lemma 4.4.2. *For* π *fixed and* $m = m(\pi), t_0 := 0, t_{m+1} := t$, *and* $1 \leq h \leq m$,*let*

$$\psi_h := \prod_{\delta_{i \leq h} = 0,1} 1_{x_{k_i}(t_i) \sim x_{l_i}(t_i)} \prod_{\delta_{i \leq h} = 0} 1_{T_i} \prod_{\delta_{i \leq h} = 2} 1_{R_i} \tag{4.4.3}$$

$$\phi_h := \prod_{\delta_{i > h} = 0,1} \frac{1}{\epsilon^{-2}(t_i - t_{i-1})^{1/2} + 1} \prod_{\delta_{i > h} = 0} \frac{1}{\epsilon^{-2}(t_{i+1} - t_i)^{1/2} + 1} \tag{4.4.4}$$

Defining $\psi_0 = \phi_h = 1$, *and recalling that* $\prod_\emptyset := 1$, *we have that* $\exists c : \forall h \leq m$:

$$\phi_h \mathbf{E}_\epsilon \psi_h \leq c \phi_{h-1} \mathbf{E}_\epsilon \psi_{h-1} \tag{4.4.5}$$

and in particular, $\mathbf{E}_\epsilon \psi_h \le c' \prod_{1\le i\le h} \frac{\phi_{i-1}}{\phi_i} =: c' g_{1,\dots,h}(t_1,\dots,t_h)$ *for some constant* c'*, where*

$$g_{1,\dots,h}(t_1,\dots,t_h) := \prod_{|\delta_i=0|} \frac{1}{[\epsilon^{-2}(t_{i+1}-t_i)]^{1/2}+1} \frac{1}{[\epsilon^{-2}(t_i-t_{i-1})]^{1/2}+1} \prod_{|\delta_i=1|} \frac{1}{[\epsilon^{-2}(t_i-t_{i-1})]^{1/2}+1}. \tag{4.4.6}$$

Proof. For simplicity, note that

$$\psi_h = \psi_{h-1}\left[1_{x_{k_h}(t_h)\sim x_{l_h}(t_h)}\left(1_{T_h,\delta_h=0}+1_{\delta_h=1}\right)+1_{R_h,\delta_h=2}\right]. \tag{4.4.7}$$

Let $\mathcal{F}_t := \sigma-$algebra generated by the *A/P-process* in $[0,t)$, and consider first the case $\delta_h = 0$. Then we have $\psi_h^{\delta_h=0} \equiv \psi_h = \psi_{h-1}1_{x_{k_h}(t_h)\sim x_{l_h}(t_h)}1_{T_h}$ for $h \le m$.

$$\Rightarrow \mathbf{E}_\epsilon\psi_h = \mathbf{E}_\epsilon \mathbf{E}_{\mathcal{F}_{t_h}}\left[\psi_{h-1}1_{x_{k_h}(t_h^-)\sim x_{l_h}(t_h^-)}1_{T_h}\right] = \mathbf{E}_\epsilon\left[\psi_{h-1}1_{x_{k_h}(t_h^-)\sim x_{l_h}(t_h^-)}\mathbf{E}_{\mathcal{F}_{t_h}}1_{T_h}\right], \tag{4.4.8}$$

since $T_h \notin \mathcal{F}_{t_h}$. Note that since we are conditioning on the σ-algebra with respect to the interval $[0,t)$ with respect to the marks process and $x(s)$, the stirring process, we have the times t_h^- above. Recalling (4.1.7), since we are conditioning on $\mathcal{F}_{t_h}$, we may consider the configuration at the time t_h as the initial configuration and bound this term as follows

$$\mathbf{E}_{\mathcal{F}_{t_h}}1_{T_h} \le \frac{c}{(\epsilon^{-2}\frac{\triangle t_h}{2})^{1/2}+1} = \frac{2^{1/2}c}{(\epsilon^{-2}(\triangle t_h))^{1/2}+2^{1/2}} \le \frac{c'}{(\epsilon^{-2}(\triangle t_h))^{1/2}+1}. \tag{4.4.9}$$

For simplicity, let $\mathcal{F}^+_{t_{h-1}} := \mathcal{F}_{t_{h-1}+\frac{1}{2}\triangle t_{h-1}}$. Again by conditioning on this $\sigma-$algebra,

$$\mathbf{E}_\epsilon\psi_{h-1}1_{x_{k_h}(t_h^-)\sim x_{l_h}(t_h^-)} = \mathbf{E}_\epsilon\mathbf{E}_{\mathcal{F}^+_{t_{h-1}}}\left[\psi_{h-1}1_{x_{k_h}(t_h^-)\sim x_{l_h}(t_h^-)}\right] = \mathbf{E}_\epsilon\left[\psi_{h-1}\mathbf{E}_{\mathcal{F}^+_{t_{h-1}}}1_{x_{k_h}(t_h^-)\sim x_{l_h}(t_h^-)}\right], \tag{4.4.10}$$

where we conditioned on the time $t_{h-1}+\frac{1}{2}\triangle t_{h-1}$ because on this time we have $\psi_{h-1} \in \mathcal{F}^+_{t_{h-1}}$ (from the definition of ψ and T_{h-1}). And by Proposition 4.1.8 we have

$$\mathbf{E}_{\mathcal{F}^+_{t_{h-1}}}1_{x_{k_h}(t_h^-)\sim x_{l_h}(t_h^-)} \le \frac{c}{[\epsilon^{-2}(t_h-t_{h-1})]^{1/2}+1}, \tag{4.4.11}$$

where to see this, one needs only to use the loss of memory property of the Markov processes. Collecting the terms we get

$$\mathbf{E}_\epsilon\psi_h \le \frac{c}{(\epsilon^{-2}\triangle t_h)^{1/2}+1}\frac{c}{(\epsilon^{-2}\triangle t_{h-1})^{1/2}+1}\mathbf{E}_\epsilon\psi_{h-1}. \tag{4.4.12}$$

Thus, recalling

$$\phi_h = \prod_{\delta_{i>h}=0,1} \frac{c}{(\epsilon^{-2}\triangle t_{i-1})^{1/2}+1} \prod_{\delta_{i>h}=0} \frac{c}{(\epsilon^{-2}\triangle t_i)^{1/2}+1} = \phi_{h-1} \frac{c}{(\epsilon^{-2}\triangle t_{h-1})^{1/2}+1} \frac{c}{(\epsilon^{-2}\triangle t_h)^{1/2}+1} \tag{4.4.13}$$

we have the desired result. Now, for $\delta_h = 1$ we have

$$\psi_h^{\delta_h=1} = \psi_{h-1} 1_{x_{k_h}(t_h^-)\sim x_{l_h}(t_h^-)}. \tag{4.4.14}$$

To get the result in the statement, simply take expectations and proceed as in (4.4.10). For $\delta_h = 2$, simply bound $1_{R_h} \le 1$ and we have

$$\psi_h^{\delta_h=2} = \psi_{h-1} 1_{R_h} \le \psi_{h-1}. \tag{4.4.15}$$

To get the quantity $g_{1,\dots,h}$, compute the first ratio ϕ_{h-1}/ψ_h then proceed by induction just as if solving a geometric recursion. □

Proof of Proposition 4.4.1. From Lemma 4.4.2 since the times do *not* cluster to t, we have

$$w'_\pi(\underline{x}, t) \le c \int_0^t dt_1 \cdots \int_{t_{m-1}}^{t-\triangle} dt_m f_{1,\dots,m}(t_1,\dots,t_m) \tag{4.4.16}$$

where

$$\begin{aligned} f_{1,\dots,m}(t_1,\dots,t_m) := \prod_{\delta_i=1} \frac{\epsilon^{-2}}{[\epsilon^{-2}(t-t_i)]^{1-\xi}+1} \frac{1}{[\epsilon^{-2}(t_i-t_{i-1})]^{1/2}+1} \\ \prod_{\delta_i=0} \frac{\epsilon^{-2}}{[\epsilon^{-2}(t-t_i)]^{1/2-\xi}+1} \frac{1}{[\epsilon^{-2}(t_i-t_{i-1})]^{1/2}+1} \frac{1}{[\epsilon^{-2}(t_{i+1}-t_i)]^{1/2}+1} \\ \prod_{\delta_i=2} \frac{\epsilon^{\theta-2}}{[\epsilon^{-2}(t_i-t_{i-1})]^{p_i/2}+1} \end{aligned} \tag{4.4.17}$$

Note that $\forall i \quad t-t_i > \triangle$, since $\triangle < t-t_m < t-t_{m-1} < \cdots < t$. We can bound the terms in (4.4.17) as follows. For the coefficient associated to $\delta_i = 1$

$$\frac{\epsilon^{-2}}{[\epsilon^{-2}(t-t_i)]^{1-\xi}+1} < \frac{\epsilon^{-2}}{[\epsilon^{-2}\triangle]^{1-\xi}+1} \le \epsilon^{-2\xi}\triangle^{\xi-1}, \quad \frac{1}{[\epsilon^{-2}(t_{i+1}-t_i)]^{1/2}+1} \le \frac{\epsilon}{(t_{i+1}-t_i)^{1/2}}, \tag{4.4.18}$$

for $\delta_i = 0$ we have

$$\frac{\epsilon^{-2}}{[\epsilon^{-2}(t-t_i)]^{1/2-\xi}+1} \le \frac{\epsilon^{-2}}{[\epsilon^{-2}\triangle]^{1/2-\xi}+1} \le \epsilon^{-2\xi}\triangle^{\xi-1/2}\epsilon^{-1}. \tag{4.4.19}$$

For $\delta_i = 2$, we break in two cases. First, let $\delta_{i-1} \ne 0$. Then, since $p_i/2 \ge 1 > 1-\xi$, $\forall \xi > 0$, by the same arguments as above we have

$$\frac{\epsilon^{\theta-2}}{[\epsilon^{-2}(t_i-t_{i-1})]^{p_i/2}+1} 1_{\delta_{i-1}\ne 0} \le 1_{p_i\ge 2,\delta_{i-1}\ne 0} \frac{\epsilon^{-2\xi}\epsilon^{\theta}}{(t_i-t_{i-1})^{1-\xi}} + 1_{p_i=1,\delta_{i-1}\ne 0} \frac{\epsilon^{-2\xi}\epsilon^{\theta-1}}{(t_i-t_{i-1})^{1/2-\xi}}. \tag{4.4.20}$$

For both $p_i = 1$ and $\delta_{i-1} = 0$ we can bound by $\frac{\epsilon^{-2\xi}\epsilon^{\theta-1}}{(t_i-t_{i-1})^{1/2-\xi}}$.

Thus we have the following bound:

$$
\begin{aligned}
&f_{1,\dots,m}(t_1,\dots,t_m) \le \prod_{\delta_i=1}[\epsilon^{-2}\triangle]^{\xi}\triangle^{-1}\frac{\epsilon}{(t_i-t_{i-1})^{1/2}} \\
&\prod_{\delta_i=0}[\epsilon^{-2}\triangle]^{\xi}\triangle^{-1/2}\epsilon^{-1}\frac{\epsilon}{(t_{i+1}-t_i)^{1/2}}\frac{\epsilon}{(t_i-t_{i-1})^{1/2}} \\
&\prod_{\delta_i=2}\left(1_{\delta_{i-1}\neq 0, p_1\ge 2}\frac{\epsilon^{-2\xi}\epsilon^{\theta}}{(t_i-t_{i-1})^{1-\xi}} + (1_{p_i\ge 2,\delta_{i-1}=0}+1_{p_i=1})\frac{\epsilon^{-2\xi}\epsilon^{\theta-1}}{(t_i-t_{i-1})^{1/2-\xi}}\right).
\end{aligned}
\tag{4.4.21}
$$

Looking only at the ξ terms, note that if did not have the terms $\epsilon^{-2}\triangle$, then we could bound the ξ terms uniformly in M. In this way, now we take $\triangle = \epsilon^a \wedge \frac{t}{M+1}$, with $a > 0$, thus bounding $\triangle^{\xi}$ by one and bounding the products of the $\epsilon^{-2\xi}$ uniformly in M, and we have that the expression in the previous display is bounded from above by $\epsilon^{-2M\xi}C_\theta(\epsilon,\triangle)\hat{f}_{1,\dots,m}(t_1,\dots,t_m)$, where

$$
\begin{aligned}
C_\theta(\epsilon,\triangle) &:= \epsilon^{|\delta_i=1|+|\delta_i=0|+\theta(|\delta_i=2,\delta_{i-1}\neq 0,p_i\ge 2|)-(1-\theta)(|\delta_i=2,\delta_{i-1}=0,p_i\ge 2|+|\delta_i=2,p_i=1|)} \\
&\quad (\triangle^{-1})^{|\delta_i=1|+\frac{1}{2}|\delta_i=0|}
\end{aligned}
\tag{4.4.22}
$$

and

$$
\begin{aligned}
&\hat{f}_{1,\dots,m}(t_1,\dots,t_m) := \prod_{\delta_i=1}\frac{1}{(t_i-t_{i-1})^{1/2-\xi}}\prod_{\delta_i=2}\frac{1}{(t_i-t_{i-1})^{q_i-\xi}} \\
&\prod_{\delta_i=0}\frac{1}{(t_{i+1}-t_i)^{1/2}}\frac{1}{(t_i-t_{i-1})^{1/2-\xi}},
\end{aligned}
\tag{4.4.23}
$$

where for simplicity we defined $q_i = 1$, if $\delta_{i-1} \neq 0 \wedge p_i \ge 2$, and $q_i = 1/2$, if $(p_i \ge 2 \wedge \delta_{i-1} = 0) \vee p_i = 1$. In this way, our bound for w'_π takes the form:

$$
\begin{aligned}
w'_\pi(\underline{x},t) &\le c\epsilon^{-2M\xi}C_\theta(\epsilon,\triangle)\int_0^t dt_1\cdots\int_{t_{m-1}}^{t-\triangle}dt_m\prod_{\delta_i=1}\frac{1}{(t_i-t_{i-1})^{1/2-\xi}}\prod_{\delta_i=2}\frac{1}{(t_i-t_{i-1})^{q_i-\xi}} \\
&\prod_{\delta_i=0}\frac{1}{(t_{i+1}-t_i)^{1/2}}\frac{1}{(t_i-t_{i-1})^{1/2-\xi}},
\end{aligned}
\tag{4.4.24}
$$

and we are finally in the step mentioned in (4.3.4). Iterating the integrals, and making a change of variables, one can show that

$$
\int_0^t dt_1\cdots\int_{t_{m-1}}^{t-\triangle}dt_m\hat{f}_{1,\dots,m} = \int_0^t dt_1\cdots\int_{t_{m-1}}^{t}dt_m 1_{t_m<t-\triangle}\hat{f}_{1,\dots,m} \le ct^S \tag{4.4.25}
$$

with $S \ge \frac{1}{2}\left(|\delta_i=1| + |\delta_i=2,p_i=1| + |\delta_i=2,p_i\ge 2,\delta_{i-1}=0|\right) - \xi M$. □

4.4.2 Bounds when times cluster

We do not bound exactly as when the times do not cluster because now we may have $t - t_i < \triangle$. In this way, the factors $(t-t_i)^{-1/2}$ and $(t-t_i)^{-1}$ lead to a negative S in the final bound. Although this difficulty lies in the final step of the proof and in a different argument, in the end the problem is completely analogous to consider the case $\theta < 1$.

Definition 4.4.3 ("Old" particle set G_H)**.** Recall that $w''_{\pi,H}$ denotes the expression (4.3.3) restricted to the *last cluster* C_H, and that the times $t_1, \dots, t_m \in \mathcal{T}_H$. Recall also that $\delta_i < 2 \Rightarrow J_i = \{k_i, l_i\}$ (from the definition of skeleton (4.3.1)). Let

$$J^0_{H,i-1} := \{\{k_j, l_j\} : \quad \delta_j = 0, \quad H \le j < i\}, \tag{4.4.26}$$

that is, the set of pairs of particles that *may* die, from iterations H to $i-1$. We define the index set G_H as the iterations after H where a particle ℓ was alive on iteration $H-1$ and is set to die *only* on iteration i:

$$G_H := \{i \ge H \mid \delta_i = 0, 1, \exists \ell \in A_{H-1} \cap J_i : \ell \notin J^0_{H,i-1}\}. \tag{4.4.27}$$

Note that in the definition of G_H it is enough to consider $\delta_i = 0$ and we do not need to set $\ell \notin J^1_{H,i-1}$ (with $J^1_{\cdot}$ defined similarly), since on that case (see the definition of skeleton (4.3.1)) both particles $\{k_i, l_i\}$ would die (and so their labels), and we would have $A_{H-1} \cap J_i = \emptyset$. Moreover, let $J^+_{H,i-1}$ be the particles that are born from iterations H to $i-1$, that is:

$$J^+_{H,i-1} := \bigcup_{H \le j < i} J^+_j. \tag{4.4.28}$$

Then clearly we have that $\ell \notin J^+_{H,i}$, where ℓ is the particle on the definition of G_H. That is,

$$\ell \notin \left(J^{0,1}_{H,i-1} \cup J^+_{H,i-1}\right) =: J^{\neq \ell}, \tag{4.4.29}$$

where $J^{0,1}_{H,i-1} := J^0_{H,i-1} \cup J^1_{H,i-1}$.

Recalling the definition of the "independent process" $\underline{x}^0(s)$ (4.1.9), the auxiliary process $\underline{y}$, where particles collide and, most importantly, (4.1.17), we attributed a (random) priority list that determines which particles colide. Letting the particle ℓ have the lowest priority we guarantee that this particle behaves as a simple random walker from iterations H to $i-1$ indepedent of any quantity. In this way, we can couple it with our stirring process, getting independence in expectations to be determined, and random walk estimates. Together with Theorem 4.1.11, we can show an analogous result to the one of Lemma 4.4.2.

Lemma 4.4.4. *$\exists c$ and $\forall k, \exists c'$ such that $\forall H$ and $t_1, \dots, t_m \in \mathcal{T}_H$; and $\forall h : H \le h \le m$ we have*

$$\phi^*_h \mathbf{E}_\epsilon \psi_h \le c \phi^*_{h-1} \mathbf{E}_\epsilon \psi_{h-1} + c'(\epsilon^{-2} \triangle)^{-k}, \tag{4.4.30}$$

where ψ_h is the same as in Lemma 4.4.2, and for $h \ge H$:

$$\phi^*_h = \prod_{i > h : i \in G_H} \frac{1}{[\epsilon^{-2} \triangle]^{1/4 - \xi} + 1} \prod_{i > h : \delta_i = 0} \frac{1}{[\epsilon^{-2}(t_{i+1} - t_i)]^{1/2} + 1} \tag{4.4.31}$$

where for $h < H$ we have $\phi^*_h := \phi_h$ as in (4.4.2).

Proof. We start by considering the case $h \geq H$ and $h \notin G_H$. Recall that

$$\psi_h = \psi_{h-1}\left[1_{x_{k_h}(t_h)\sim x_{l_h}(t_h)}\left(1_{T_h,\delta_h=0}+1_{\delta_h=1}\right)+1_{R_h,\delta_h=2}\right], \tag{4.4.32}$$

and the events

$$R_i := \{x_j(t_i^-) \in I^c_{u_i}, j \notin J_i\}, \qquad T_i := \{\tau_{k_i,l_i}(t_i) > (t_{i+1}+t_i)/2\}. \tag{4.4.33}$$

For $\delta_h = 0,1$ we can bound $1_{x_{k_h}(t_h)\sim x_{l_h}(t_h)} \leq 1$ and proceed to bound $\mathbf{E}_\epsilon\psi_h = \mathbf{E}_\epsilon\left[\psi_{h-1}\left(1_{T_h,\delta_h=0}+1_{\delta_h=1}\right)\right]$ as in Lemma 4.4.2 to get:

$$\mathbf{E}_\epsilon\psi_h^{\delta_h=0} \leq \mathbf{E}_\epsilon\left[\psi_{h-1}1_{T_h}\right] = \mathbf{E}_\epsilon\mathbf{E}_{\mathcal{F}_{t_h}}\left[\psi_{h-1}1_{T_h}\right] \leq \frac{c}{[\epsilon^{-2}(t_{h+1}-t_h)]^{1/2}+1}, \tag{4.4.34}$$

while $\mathbf{E}_\epsilon\psi_h^{\delta_h=1} \leq 1$. For $\delta_h = 2$ we bound $1_{R_h} \leq 1$.

Now we consider the case $h \geq H$ and $h \in G_H$. We factor: $\psi_h = \psi_{H-1}\psi^{\neq l}_{H,h}\psi^{\ell,+}_{H,h}$, where the lowerscript restricts in ψ the products over $H \leq i < h$ and the upperscript restricts to the sets in (4.4.28) and (4.4.29)

$$\psi^{\neq\ell}_{H,h} := \prod_{H\leq i<h:\delta_i=0,1} 1_{x_{l_i}(t_i)\sim x_{k_i}(t_i)} \prod_{H\leq i<h:\delta_i=0} 1_{T_i} \prod_{H\leq i<h:\delta_i=2} 1_{x_j(t_i^-)\in I^c_{u_i},\ell\neq j\notin J_i}, \tag{4.4.35}$$

and $\psi^{\ell,+}_{H,h} := \psi^{\ell}_{H,h}1_{T_h}$ where

$$\psi^{\ell}_{H,h} := 1_{x_{k_h}(t_h)\sim x_{l_h}(t_h)} \prod_{H\leq i\leq h:\delta_i=2} 1_{x_\ell(t_i^-)\in I^c_{u_i}}. \tag{4.4.36}$$

We remark that $\psi^{\neq\ell}_{H,h} \perp\!\!\!\perp x_\ell(\cdot)$. Bounding 1_{T_h} as in Lemma 4.4.2, we have

$$\mathbf{E}_\epsilon\psi_h = \mathbf{E}_\epsilon\left[\psi_{H-1}\psi^{\neq\ell}_{H,h}\psi^{\ell}_{H,h}1_{T_h}\right] \leq c\frac{\mathbf{E}_\epsilon\left[\psi_{H-1}\psi^{\neq\ell}_{H,h}\psi^{\ell}_{H,h}\right]}{[\epsilon^{-2}(t_{h+1}-t_h)]^{1/2}+1}. \tag{4.4.37}$$

Now we treat the expectation in the last inequality. Since $h \in G_H$ (hence implicit in $\psi^{\ell}_{H,h}$), we know the particle $\ell \in J_h = \{k_h, l_h\}$. Without loss of generality let $\ell = l_h$. Again, by the law of total expectation we condition to $\mathcal{F}_{t_{H-1}}$ in order to bound the other terms, thus getting:

$$\mathbf{E}_\epsilon\left[\psi_{H-1}\mathbf{E}_{\mathcal{F}_{t_{H-1}}}\left(\psi^{\neq\ell}_{H,h}\psi^{\ell}_{H,h}\right)\right]. \tag{4.4.38}$$

Recalling (4.1.17) and the definition of branching process (4.3.4) we realize and couple the process $\underline{x}^0(\cdot)$ in each interval $]t_{i,i+1}[$ with $i \geq H-1$ with "births" (with new independent particles and new labels and priorities) and "deaths" (with corresponding removal of labels) determined by π. Thus the process $\underline{x}^0$ starts from t_{H-1} with "initial" configuration $\underline{x}^* = \underline{x}(t_h)$. Giving ℓ the lowest priority we guarantee that it remains alive up to t_h (see(4.1.17) and (4.4.27)). Moreover, particle ℓ has initial position x^*_ℓ. In this way, we will write

$$\mathbf{E}^*\left(\psi^{\neq\ell}_{H,h}\psi^{\ell}_{H,h}\right) := \mathbf{E}_{\mathcal{F}_{t_{H-1}}}\left(\psi^{\neq\ell}_{H,h}\psi^{\ell}_{H,h}\right). \tag{4.4.39}$$

Recall from (4.1.17) that we have

$$\big(\sigma(\ell) = 1 \wedge x_\ell(0) = x_\ell^0(0)\big) \Rightarrow P_\epsilon(x_\ell(t) = x_\ell^0(t)) = 1 \quad \forall t \geq 0, \tag{4.4.40}$$

where $\sigma = 1$ is the *lowest* priority. In this way, if the particle ℓ has the same walk both in the *independent process* and the original process, any stirring particle position x_s, with $m \neq \ell$ and $t_{H-1} \leq s \leq t$ is then a function of only the independent processes $x_k^0(s), t_{H-1} \leq s \leq t$ and $k \neq \ell$. Thus, expression (4.4.39) becomes an expectation of independent particles, since $\forall i, t \geq 0, \quad x_i(t)$ is *completely* determined by $y_i(s), s \in [0,t]$ and $j : \sigma(j) \leq \sigma(i)$ (recall the other items in (4.1.17)). In order to exploit Theorem 4.1.11, we decompose the identity function as $1 = \chi + (1-\chi)$, where

$$\chi = 1_{\left|x_\ell(t_h) - x_\ell^0(t_h)\right| \leq (\epsilon^2 \triangle)^{1/4+\xi}} \prod_{H \leq i \leq h : \delta_i = 2} 1_{\left|x_\ell(t_i) - x_\ell^0(t_i)\right| \leq (\epsilon^2 \triangle)^{1/4+\xi}}. \tag{4.4.41}$$

Now define the event $\chi_{H,h}^\ell := \{\exists i \in (\delta_{[H,h]} = 2) : \left|x_\ell(t_i) - x_\ell^0(t_i)\right| > (\epsilon^2 \triangle)^{1/4+\xi}\}$. Then $\{\chi = 0\} = \bigcup_{H \leq j \leq h} \chi_j^\ell$. In this way, we have

$$P^*(\chi = 0) \leq \sum_{i \in (\delta_{[H,h]} = 2)} P^*(\left|x_\ell(t_i) - x_\ell^0(t_i)\right| > (\epsilon^2 \triangle)^{1/4+\xi}) \tag{4.4.42}$$

and by Theorem 4.1.11, this is bounded by $(h - H + 1)c(\epsilon^{-2}\triangle)^{-k} = c'(\epsilon^{-2}\triangle)^{-k}$ for any $k > 1$, where we used the bound $(t_i - t_{H-1})^{-k} \leq (i - (H-1))^{-k}\triangle^{-k} \leq \triangle^{-k}$. Using this, we have

$$\begin{aligned} \mathbf{E}^*\left(\psi_{H,h}^{\neq \ell}\psi_{H,h}^\ell\right) &= \mathbf{E}^*\left(\psi_{H,h}^{\neq \ell}\psi_{H,h}^\ell(1-\chi)\right) + \mathbf{E}^*\left(\psi_{H,h}^{\neq \ell}\psi_{H,h}^\ell\chi\right) \\ &\leq c'(\epsilon^{-2}\triangle)^{-k} + \mathbf{E}^*\left(\psi_{H,h}^{\neq \ell}\psi_{H,h}^\ell\chi\right), \end{aligned} \tag{4.4.43}$$

where we bounded $\psi_{H,h}^{\neq \ell}, \psi_{H,h}^\ell \leq 1$ in the first expectation. Now we only need to worry with the last term in the previous display. With a clear abuse of notation we will write $\{\left|x_\ell(t_i^-) - I_{u_i}\right| > 0\} = \{x_\ell(t_i^-) \notin I_{u_i}\}$. In this way, define the set $F_i^> := \{x \in \Lambda_N : |x - I_{u_i}| > (\epsilon^{-2}\triangle)^{1/4+\xi}\}$, that is, all the sites that are *not* in a radius of $(\epsilon^{-2}\triangle)^{1/4+\xi}$ from any point of I_{u_i}. Define also $(\delta_{H,h} = 2)^{\ell^0 \notin F} := \{k \in (\delta_{[H,h]} = 2) : x_\ell^0(t_k) \notin F_k^>\}$ *i.e.,* all the iterations $H \leq k \leq h$ where $\delta_k = 2$ and the *independent* particles are in a radius smaller or equal to $(\epsilon^{-2}\triangle)^{1/4+\xi}$ from some point of I_{u_i}. Finally, define

$$\omega = \begin{cases} i & , i = \min(\delta_{H,h} = 2)^{\ell^0 \notin F} \\ h & , (\delta_{H,h} = 2)^{\ell^0 \notin F} = \emptyset. \end{cases} \tag{4.4.44}$$

The variable above gives the first step when an independent particle is in the aforementioned radius, if such event ever happens, or takes the value h if that never happens. It is clear that $\sum_{i \in (\delta_{[H,h]} = 2)} 1_{\omega = i} = 1$.

We consider first the case $\omega < h$. In this case we have

$$\mathbf{E}^*\left(\psi_{H,h}^{\neq \ell}\psi_{H,h}^\ell\chi(\sum_{i \in (\delta_{[H,h]} = 2)} 1_{\omega = i})\right) = \sum_{i \in (\delta_{[H,h]} = 2)} \mathbf{E}^*\left(\psi_{H,h}^{\neq \ell}\psi_{H,h}^\ell\chi 1_{\omega = i}\right). \tag{4.4.45}$$

Looking at the summand in the expression above, we can bound $\psi^{\ell}_{H,h}\chi \leq 1$, thus getting

$$\mathbf{E}^* \left(\psi^{\neq \ell}_{H,h}\psi^{\ell}_{H,h}\chi 1_{\omega=i}\right) \leq \mathbf{E}^* \left(\psi^{\neq \ell}_{H,h} 1_{\omega=i}\right). \tag{4.4.46}$$

Treating the indicator function, we get:

$$\{\omega = i\} = \{i = \min(\delta_{H,h} = 2)^{\ell^0 \notin F}\} = \bigcap_{k\in(\delta_{H,h}=2)} \{k : x^0_\ell(t_k) \notin F_k, k \geq i\}. \tag{4.4.47}$$

Thus, $1_{\omega=i} \leq 1_{x^0_\ell(t_i)\notin F_i}$. Moreover, $\psi^{\neq\ell}_{H,h} \perp\!\!\!\perp x^0_\ell \Rightarrow \psi^{\neq\ell}_{H,h} \perp\!\!\!\perp f(x^0_\ell)$ and we can take the expectation of the product as the product of expectations in (4.4.46):

$$\mathbf{E}^*\psi^{\neq\ell}_{H,h}P^*(\omega = i) \leq \mathbf{E}^*\psi^{\neq\ell}_{H,h}P^*(x^0_\ell(t_i) \notin F_i), \tag{4.4.48}$$

and one can bound $P^*(x^0_\ell(t_i) \notin F_i) \leq \frac{c}{(\epsilon^{-2}\triangle)^{1/4-\xi}+1}$. Now we consider the case $\omega = h$. Recall that we want to bound $\sum_{i\in(\delta_{[H,h]}=2)} \mathbf{E}^* \left(\psi^{\neq\ell}_{H,h}\psi^{\ell}_{H,h}\chi 1_{\omega=i}\right)$, and, in particular, the summand $\mathbf{E}^* \left(\psi^{\neq\ell}_{H,h}\psi^{\ell}_{H,h}\chi 1_{\omega=i}\right)$, where $\psi^{\ell}_{H,h} = 1_{x_{k_h}(t_h)\sim x_{l_h}(t_h)} \prod_{H\leq i\leq h:\delta_i=2} 1_{x_\ell(t_i^-)\in I^c_{u_i}}$. Note that $\forall i \in (\delta_{[H,h]} = 2)$ with $i < h$, by the definition of ω and χ we have $1_{x_\ell(t_i)\notin I_{u_i}}\chi 1_{\omega=h} = \chi 1_{\omega=h}$.

Now let us focus in the term $1_{x_{k_h}(t_h)\sim x_{l_h}(t_h)}\chi 1_{\omega=h}$. Note that,

$$|x_\ell(t_h) - x_{k_h}(t_h)| = \left|x^0_\ell(t_h) - x_{k_h}(t_h) - (x^0_\ell(t_h) - x_\ell(t_h))\right| \geq \left|x^0_\ell(t_h) - x_{k_h}(t_h)\right| - \left|(x^0_\ell(t_h) - x_\ell(t_h))\right|. \tag{4.4.49}$$

If $\left|(x^0_\ell(t_h) - x_\ell(t_h))\right| \leq (\epsilon^{-2}\triangle)^{1/4+\xi}$ and $x_\ell(t_h) \sim x_{k_h}(t_h)$, the inequality above takes the form

$$\left|x^0_\ell(t_h) - x_{k_h}(t_h)\right| \leq 1 + (\epsilon^{-2}\triangle)^{1/4+\xi} \tag{4.4.50}$$

and we can bound $1_{x_{k_h}(t_h)\sim x_{l_h}(t_h)}\chi 1_{\omega=h} \leq 1_{\left|x^0_\ell(t_h)-x_{k_h}(t_h)\right|\leq 1+(\epsilon^{-2}\triangle)^{1/4+\xi}}\chi 1_{\omega=h}$. Bounding $\chi 1_{\omega=h} \leq 1$ we get

$$\mathbf{E}^* \left(\psi^{\neq\ell}_{H,h}\psi^{\ell}_{H,h}\chi 1_{\omega=h}\right) \leq \mathbf{E}^* \left(\psi^{\neq\ell}_{H,h}\psi^{\ell}_{H,h} 1_{\left|x^0_\ell(t_h)-x_{k_h}(t_h)\right|\leq 1+(\epsilon^{-2}\triangle)^{1/4+\xi}}\right). \tag{4.4.51}$$

Now we can follow x_ℓ by conditioning on the $\sigma-$algebra generated by all the *other* variables $x^0_j, j \neq \ell$, that is $\mathcal{F}^{\neq,\ell}$. This way, and recalling that we have already bounded $\psi^{\ell}_{H,h}$, we have $\psi^{\neq\ell}_{H,h} \in \mathcal{F}_{\neq,\ell}$ and

$$\mathbf{E}^* \left(\psi^{\neq\ell}_{H,h}\psi^{\ell}_{H,h} 1_{\left|x^0_\ell(t_h)-x_{k_h}(t_h)\right|\leq 1+(\epsilon^{-2}\triangle)^{1/4+\xi}}\right) \leq \mathbf{E}^* \left(\psi^{\neq\ell}_{H,h}\mathbf{E}_{\mathcal{F}_{\neq,\ell}} 1_{\left|x^0_\ell(t_h)-x_{k_h}(t_h)\right|\leq 1+(\epsilon^{-2}\triangle)^{1/4+\xi}}\right). \tag{4.4.52}$$

Since, under $\mathcal{F}_{\neq,\ell}$, the particle x^0_ℓ is a simple random walker we can bound

$$\mathbf{E}_{\mathcal{F}_{\neq,\ell}} \left(1_{\left|x^0_\ell(t_h)-x_{k_h}(t_h)\right|\leq 1+(\epsilon^{-2}\triangle)^{1/4+\xi}}\right) \leq \frac{c}{(\epsilon^{-2}\triangle)^{1/4-\xi}+1}. \tag{4.4.53}$$

Noticing that $\mathbf{E}_\epsilon(\psi_{H-1}\psi^{\neq\ell}_{H,h}) = \mathbf{E}_\epsilon\psi_{h-1}$, one can collect all the estimates to get the final bound in the statement.

□

Now we have all the ingredients to prove an analogous bound of Proposition 4.4.1. The arguments for the proof are analogous, except for one step. Recall that in Lemma 4.4.4 we have the term $c'(\epsilon^{-2}\triangle)^{-k}$ for any k. In this way, in the bound above we chose k large enough for this term to be negligible. Then, one may proceed analagously to Lemma 4.4.4 to show the bound below.

Proposition 4.4.5. *$\forall\xi > 0\exists\, c : \forall\pi$ such that $m = m(\pi) \leq M, \underline{x} : |\underline{x}| = n, \epsilon > 0, t \leq \epsilon^{\beta^*}$ we have*

$$\begin{aligned} w''_{\pi,H}(\underline{x},t) \leq & \left\{ c(\epsilon^2 t)^{-M\xi}\triangle^{-S_1(H-1)}\epsilon^{S_2(H-1)}t^{S_3(H-1)}\epsilon^{-S_4^{(\theta)}(H-1)} \right\} \times \\ & \times \left\{ c(\epsilon^2\triangle)^{-M\xi}(\epsilon^{-2}\triangle)^{-\frac{1}{4}|G_H|}\triangle^{\frac{1}{2}\left|\delta_{i\geq H}=2\right|}\epsilon^{(\theta-1)\left|\delta_{i\geq H}=2\right|} \right\}, \end{aligned} \tag{4.4.54}$$

with the $S_k(\cdot)$ exponents as in Proposition 4.4.1:

$$\begin{aligned} S_1(H-1) &= |\delta_i = 1| + \frac{1}{2}|\delta_i = 0|, \\ S_2(H-1) &= |\delta_i = 0,1| + |\delta_i = 2, p_i \geq 2, \delta_{i-1} \neq 0|, \\ S_3(H-1) &= \frac{1}{2}|\delta_i = 1| + |\delta_i = 2, p_i = 1| + |\delta_i = 2, p_i \geq 2, \delta_{i-1} = 0|, \\ S_4^{(\theta)}(H-1) &= (1-\theta)\left(|\delta_i = 2, \delta_{i-1} > 0, p_i \geq 2| + |\delta_i = 2, \delta_{i-1} = 0, p_i \geq 2| + |\delta_i = 2, p_i = 1|\right), \end{aligned} \tag{4.4.55}$$

with $i \leq H-1$.

4.5 Proof of the v-functions estimate

Now the main idea is to play with the exponents in such a way that the desired bound arises. If $m = M$, then the idea is to bound the terms in order to get an expression slightly larger than M in the exponent. In this way, we may bound the exponent from below by M and then choose M accordingly. If $m < M$, then the arguments are different. The idea is to bound directly in terms of n, thus the arguments will rely in inequalities relating the final balance of particles. In order to give the reader some intuition for the following computations, we will derive some inequalities that will be useful for the following proofs.

- Let $m(\pi) = M$ and w'_π:

$$M = |\delta_i = 0| + |\delta_i = 1| + |\delta_i = 2| \tag{4.5.1}$$

$$M \leq |\delta_i = 1| + |\delta_i = 0| + |\delta_i = 2, p_i \geq 2| + |\delta_i = 2, p_i = 1| \tag{4.5.2}$$

$$n - (|\delta_i = 1| + |\delta_i = 0|) + (K-1)|\delta_i = 2, p_i \geq 2| \geq 0 \tag{4.5.3}$$

The first equality is the definition of M. For the second, we simply break $|\delta_i = 2|$ in two. For the last, note that for $m(\pi) = M$ we did not kill all the particles. Thus, we have a positive balance of particles: $n - deaths + birhts \geq 0$. For $\delta_i = 0,1$, the total number of deaths is $2|\delta_i = 1| + |\delta_i = 0|$. For $\delta_i = 2$ we know that $p_i = 1 \Rightarrow \left|J_i^+\right| > 0$, thus we have $b_1|\delta_i = 2, p_i = 1|$ births under these conditions (for some $b_1 > 0$). Trivially, we have under these conditions $|\delta_i = 2, p_i = 1|$ deaths. Recall that if $p_i \geq 2$ then J_i^+ might be empty. Thus, we have *at least* $2|\delta_i = 2, p_i \geq 2|$ deaths (exactly $d|\delta_i = 2, p_i \geq 2|$ for some $d \geq 2$) and $b_2|\delta_i = 2, p_i \geq 2|$ births, for some $b_2 \geq 0$. Since we may have *at most*, K births per step, we know that $0 \leq b_1, b_2 \leq K$. Finally, we conclude that

$$n - 2|\delta_i = 1| - |\delta_i = 0| + (b_1 - 1)|\delta_i = 2, p_i = 1| + (b_2 - d)|\delta_i = 2, p_i \geq 2| \geq 0 \tag{4.5.4}$$

For some constants b_1, b_2, d.

Since M is fixed, the number of skeletons is *finite*. In this way, recalling that $|v(\underline{x}, t)| \leq c \sum_\pi w_\pi(\underline{x}, t)$, it suffices to show the bounds

$$\max_\pi w'_\pi(\underline{x}, t) \leq c(\epsilon^{-2}t)^{-c^* n}, \quad \max_{\pi, H} w''_{\pi, H}(\underline{x}, t) \leq c(\epsilon^{-2}t)^{-c^* n} \tag{4.5.5}$$

Recalling Proposition 4.4.1, we have:

$$w'_\pi(\underline{x}, t) \leq c(\epsilon^2 t)^{-\xi M} \triangle^{-S_1(m)} \epsilon^{S_2(m)} t^{S_3(m)} \epsilon^{-S_4^{(\theta)}(m)} \tag{4.5.6}$$

- $m(\pi) = M, w'_\pi, \triangle = \epsilon^a, \theta = 1.$

Note that since $t \leq 1$, we have $t^{S_3(m)} \leq t^{\frac{1}{2}|\delta_2 = 2, p_i = 1|}$. Rearranging (4.5.6):

$$w'_\pi(\underline{x}, t) \leq c(\epsilon^2 t)^{-M\xi} [\epsilon \triangle^{-1}]^{|\delta_i = 1|} \triangle^{-\frac{1}{2}|\delta_i = 0|} \epsilon^{|\delta_i = 0| + |\delta_i = 2, p_i \geq 2, \delta_{i-1} \neq 0|} t^{\frac{1}{2}|\delta_i = 2, p_i = 1|}. \tag{4.5.7}$$

Since all the terms are of order of ϵ, and $a < 1$ on the definition of $\triangle$, the idea is to group $\epsilon, \triangle$ as $[\epsilon \triangle^{-\frac{1}{2}}]^k$ for some k function of δ_i not present in the t and $\epsilon \triangle^{-1}$ terms. In this way, we can bound the exponents to something "close" to M:

$$\begin{aligned} &\triangle^{-\frac{1}{2}|\delta_i = 0|} \epsilon^{|\delta_i = 0| + |\delta_i = 2, p_i \geq 2, \delta_{i-1} \neq 0|} \leq \triangle^{-\frac{1}{2}|\delta_i = 0|} \epsilon^{|\delta_i = 0| + |\delta_i = 2, p_i \geq 2|} = [\epsilon \triangle^{-\frac{1}{2}}]^{\frac{1}{2}(|\delta_i = 0| + |\delta_i = 2, p_i \geq 2|)} \\ &\epsilon^{\frac{1}{2}|\delta_i = 0| + |\delta_i = 2, p_i \geq 2, \delta_{i-1} \neq 0| - \frac{1}{2}|\delta_i = 2, p_i \geq 2|} \triangle^{\frac{1}{4}|\delta_i = 2, p_i \geq 2| + \frac{1}{4}|\delta_i = 0|}. \end{aligned} \tag{4.5.8}$$

Now note that $\triangle^{\frac{1}{4}|\delta_i = 2, p_i \geq 2| + \frac{1}{4}|\delta_i = 0|} \leq 1$ and

$$\epsilon^{\frac{1}{2}|\delta_i = 0| + |\delta_i = 2, p_i \geq 2, \delta_{i-1} \neq 0| - \frac{1}{2}|\delta_i = 2, p_i \geq 2|} \leq \epsilon^{\frac{1}{2}(|\delta_i = 0| + |\delta_i = 2, p_i \geq 2, \delta_{i-1} \neq 0| - |\delta_i = 2, p_i \geq 2|)}, \tag{4.5.9}$$

where we can bound the last expression by 1 with the following argument. For sets A, B, C we have

$$|A \cap B| = |A \cap B \cap (C \cup C^c)| \leq |A \cap B \cap C| + |A \cap B \cap C^c|. \tag{4.5.10}$$

Letting $A \equiv \{\delta_i = 2\}, B \equiv \{p_i \geq 2\}$ and $C \equiv \{\delta_{i-1} > 0\}$ we have

$$|\delta_i = 0| + |\delta_i = 2, p_i \geq 2, \delta_{i-1} > 0| \geq |\delta_i = 2, p_i \geq 2|, \tag{4.5.11}$$

and we get $w'_\pi(\underline{x}, t) \leq c(\epsilon^2 t)^{-M\xi} [\epsilon \triangle^{-1}]^{|\delta_i = 1|} [\epsilon \triangle^{-\frac{1}{2}}]^{\frac{1}{2}(|\delta_i = 0| + |\delta_i = 2, p_i \geq 2|)} t^{\frac{1}{2}|\delta_i = 2, p_i = 1|}$. Since $t \leq \epsilon^{\beta^*}$ we can set $\epsilon^b := \max\{\epsilon\triangle^{-1}, [\epsilon\triangle^{-\frac{1}{2}}]^{\frac{1}{2}}, \epsilon^{\frac{1}{2}\beta^*}\}$ to get

$$w'_\pi(\underline{x}, t) \leq c(\epsilon^{-2}t)^{-M\xi}[\epsilon^b]^{|\delta_i=1|+|\delta_i=0|+|\delta_i=2,p_i\geq 2|+|\delta_i=2,p_i=1|}. \tag{4.5.12}$$

By the same argument as (4.5.10) the exponent on ϵ^b is bounded from below by M. Applying the bound on f we have $w'_\pi(\underline{x}, t) \leq c(\epsilon^{2+\beta^*})^{-M\xi}\epsilon^{Mb}$, and we can choose $bM \geq 2n$ to get $w'_\pi(\underline{x}, t) \leq c'\epsilon^n$. Taking ξ small enough and $c^* < 1/2$ the proof is done.

- $m(\pi) = M, w'_\pi, \triangle = \epsilon^a, \theta > 1.$

Note that for this case $-S_4^\theta(m) \geq 0$. Instead of bounding $\epsilon^{-S_4^\theta(m)} \leq 1$ we shall do a little better. Recalling (4.5.1) note that

$$|\delta_i = 0| + |\delta_i = 1| + |\delta_i = 2, p_i \geq 2| \leq M. \tag{4.5.13}$$

In this way, we have that

$$|\delta_i = 2, p_i \geq 2| \geq \frac{|\delta_i = 1| + |\delta_i = 0|}{K-1} - \frac{n}{K-1}. \tag{4.5.14}$$

replacing $|\delta_i = 2, p_i \geq 2|$ in (4.5.13) we get

$$(|\delta_i = 1| + |\delta_i = 0|)(1 + \frac{1}{K-1}) \leq M + \frac{n}{K-1} \Leftrightarrow |\delta_i = 1| + |\delta_i = 0| \leq M\frac{K-1}{K} + \frac{n}{K}. \tag{4.5.15}$$

Thus, we can relate $\delta_i = 2$ and M, K, n through a nice bound:

$$|\delta_i = 2| = M - (|\delta_i = 0| + |\delta_i = 1|) \geq M(1 - \frac{K-1}{K}) - \frac{n}{K} \geq |\delta_i = 2| \geq \frac{M-n}{K}. \tag{4.5.16}$$

Now note that $S_4^{(\theta)}(m) \geq (1-\theta)|\delta_i = 2| \geq (\theta - 1)\frac{M-n}{K}$. This way, we can bound

$$\epsilon^{S_4^{(\theta)}(m)} \leq \epsilon^{(\theta-1)\frac{M-n}{K}}. \tag{4.5.17}$$

Proceeding exactly as for the case $\theta = 1$ and leaving the term above be still, we arrive at

$$w'_\pi(\underline{x}, t) \leq c(\epsilon^{2+\beta^*})^{-M\xi}\epsilon^{Mb+(\theta-1)\frac{M-n}{K}}. \tag{4.5.18}$$

Looking at the exponent: $Mb+(\theta-1)(M-n)/K = (M(bK+\theta-1)-(\theta-1)n)/K$, one can choose $MbK \geq n(\theta-1)$ to get $w'_\pi(\underline{x}, t) \leq c'\epsilon^{c^*\theta n}$ for ξ small enough .

- $m(\pi) = M, w'_\pi, \triangle = \frac{t}{M+1}, \theta = 1$

  Rearrange (4.5.6) to:

$$\begin{aligned} w'_\pi(\underline{x}, t) \leq c(\epsilon^2 t)^{-M\xi}[t^{1/2}]^{|\delta_i=2,p_i=1|+|\delta_i=2,p_i\geq 2,\delta_{i-1}=0|}\epsilon^{|\delta_i=2,p_i\geq 2,\delta_{i-1}\neq 0|} \\ \epsilon^{|\delta_i=0|+|\delta_i=1|}[\triangle^{-1}]^{|\delta_i=1|+\frac{1}{2}|\delta_i=0|}[t^{1/2}]^{|\delta_i=1|}. \end{aligned} \tag{4.5.19}$$

We can group and bound the last three terms as follows:

$$[\epsilon t^{-1/2}]^{|\delta_i=0|+|\delta_i=1|}[t\triangle^{-1}]^{|\delta_i=1|+\frac{1}{2}|\delta_i=0|} \leq [\epsilon t^{-1/2}]^{|\delta_i=0|+|\delta_i=1|}, \tag{4.5.20}$$

since $\triangle$ is of order of t. Thus,

$$w'_\pi(\underline{x},t) \leq c(\epsilon^2 t)^{-M\xi}[\epsilon t^{-1/2}]^{|\delta_i=0|+|\delta_i=1|}\epsilon^{|\delta_i=2,p_i\geq 2,\delta_{i-1}\neq 0|}[t^{1/2}]^{|\delta_i=2,p_i=1|+|\delta_i=2,p_i\geq 2,\delta_{i-1}=0|}. \tag{4.5.21}$$

Recall that we are considering only "small" times, $t \leq \epsilon^{\beta^*}$. This way, $\epsilon t^{-1/2} \leq \epsilon^{1-\frac{1}{2}\beta^*}$ and if $\beta^* \leq 2$ we can bound $[\epsilon t^{-1/2}]^{|\delta_i=0|+|\delta_i=1|} \leq 1$.

Remark 4.5.1. Note that $\beta^* \leq 2$ means that our bounds are good only for $t \in [\epsilon^2, \epsilon^\beta]$. For $t \leq \epsilon^2$ we bound the v-functions by 1.

With the bound mentioned above we have

$$w'_\pi(\underline{x},t) \leq c(\epsilon^2 t)^{-M\xi}\epsilon^{|\delta_i=2,p_i\geq 2,\delta_{i-1}\neq 0|}[t^{1/2}]^{|\delta_i=2,p_i=1|+|\delta_i=2,p_i\geq 2,\delta_{i-1}=0|}. \tag{4.5.22}$$

Again, note that $t \geq \epsilon^2 \Rightarrow t^{-M\xi} \leq \epsilon^{-2M\xi} \Rightarrow (\epsilon^2 t)^{-M\xi} \leq (\epsilon^2)^{-4M\xi}$. This way we have only powers of ϵ, but no $\delta_i = 0, 1$ to relate with M. Still, we can apply (4.5.16) to get powers of M, n and K:

$$w'_\pi(\underline{x},t) \leq c\epsilon^{-4M\xi}\epsilon^{|\delta_i=2,p_i\geq 2,\delta_{i-1}\neq 0|+\frac{\beta^*}{2}(|\delta_i=2,p_i=1|+|\delta_i=2,p_i\geq 2,\delta_{i-1}=0|)}. \tag{4.5.23}$$

Now note that $\epsilon^2 \leq \epsilon^{\beta^*} \Rightarrow \epsilon \leq \epsilon^{\beta^*/2}$ thus, we have $\epsilon^{|\delta_i=2,p_i\geq 2,\delta_{i-1}\neq 0|} \leq \epsilon^{\frac{1}{2}\beta^*(|\delta_i=2,p_i\geq 2,\delta_{i-1}\neq 0|)}$, and we can bound

$$\begin{aligned} w'_\pi(\underline{x},t) &\leq c\epsilon^{-4M\xi}\epsilon^{\frac{\beta^*}{2}(|\delta_i=2,p_i\geq 2,\delta_{i-1}\neq 0||\delta_i=2,p_i=1|+|\delta_i=2,p_i\geq 2,\delta_{i-1}=0|)} \\ &\leq c\epsilon^{-4M\xi}\epsilon^{\frac{\beta^*}{2}|\delta_i=2|} \leq c\epsilon^{-4M\xi}\epsilon^{\frac{\beta^*}{2K}(M-n)}. \end{aligned} \tag{4.5.24}$$

Letting $\frac{\beta^*}{2K}M \geq 3nK$ we get $w'_\pi(\underline{x},t) \leq c\epsilon^{-4M\xi}\epsilon^{2n}$. Taking ξ small enough and $c^* < 1/2$ we get the result in (4.5.5).

- $m(\pi) = M, w'_\pi, \triangle = \frac{t}{M+1}, \theta > 1$

The result follows by simply applying (4.5.17):

$$\epsilon^{S_4^{(\theta)}(m)} \leq \epsilon^{(\theta-1)\frac{M-n}{K}}, \tag{4.5.25}$$

and proceeding exactly as we did for $\theta = 1$. In the end we get

$$w'_\pi(\underline{x},t) \leq c\epsilon^{-4M\xi}\epsilon^{\frac{\beta^*}{2K}(M-n)}\epsilon^{(\theta-1)\frac{M-n}{K}} = c\epsilon^{-4M\xi}\epsilon^{\frac{\beta^*+2(\theta-1)}{2K}(M-n)} \tag{4.5.26}$$

Now we may take, for example $M > 2Kn$.

- $m(\pi) < M, w'_\pi, \triangle = \frac{t}{M+1}, \theta = 1.$

Recall that for $m(\pi) < M$ all particles have died on iteration m - there is no more v-function. In this way, we have

$$n - 2|\delta_i = 1| - |\delta_i = 0| + (b_1 - 1)|\delta_i = 2, p_i = 1| + (b_2 - d)|\delta_i = 2, p_i \geq 2| \leq 0, \tag{4.5.27}$$

and we have that

$$n \leq 2|\delta_i = 1| + |\delta_i = 0| + K|\delta_i = 2, p_i \geq 2|. \tag{4.5.28}$$

Multiplying and dividing (4.5.6) by $t^{\frac{1}{2}|\delta_i=2,p_i\geq 2,\delta_{i-1}\neq 0|}$ and rearranging the terms we have

$$\begin{aligned} w'_\pi(\underline{x}, t) \leq c(\epsilon^2 t)^{-M\xi}(\epsilon t^{-1/2})^{|\delta_i=0|+|\delta_i=1|+|\delta_i=2,\delta_{i-1}\neq 0,p_i\geq 2|} \\ t^{\frac{1}{2}(|\delta_i=2,p_i=1|+|\delta_i=2,p_i\geq 2|)} t^{|\delta_i=2,\delta_{i-1}\neq 0,p_i\geq 2|}(t\triangle^{-1})^{\frac{1}{2}|\delta_i=0|+|\delta_i=1|}. \end{aligned} \tag{4.5.29}$$

Since $\triangle$ is of the order of t, we bound the last line on the previous display by a constant. By (4.5.28) and factoring $|\delta_i = 2, p_1 \geq 2|$ as in (4.5.11) we get

$$(K+1)|\delta_i = 0| + 2|\delta_i = 1| + K|\delta_i = 2, p_i \geq 2, \delta_{i-1} > 0| \geq n. \tag{4.5.30}$$

Since

$$\begin{aligned} &|\delta_i = 0| + |\delta_i = 1| + |\delta_i = 2, \delta_{i-1} \neq 0, p_i \geq 2| \geq \\ &\geq |\delta_i = 0| + \frac{2}{K+1}|\delta_i = 1| + \frac{K}{K+1}|\delta_i = 2, \delta_{i-1} \neq 0, p_i \geq 2| \geq \frac{n}{K+1}, \end{aligned} \tag{4.5.31}$$

we automatically have the bound $w'_\pi(\underline{x}, t) \leq c(\epsilon^2 t)^{-\xi M}(\epsilon t^{-1/2})^{\frac{n}{K+1}}$, which is consistent with (4.5.5) if we have that $c^* < \frac{1}{2(1+K)}$.

- $m(\pi) < M, w'_\pi, \triangle = \epsilon^a, \theta = 1$

Bounding $t^{S_3(m)}$ by one and rearranging the terms we arrive at

$$\begin{aligned} w'_\pi(\underline{x}, t) &\leq c(\epsilon^2 t)^{-\xi M}(\epsilon\triangle^{-1})^{|\delta_i=1|}(\epsilon\triangle^{-\frac{1}{2}})^{|\delta_i=0|}\epsilon^{|\delta_i=2,p_i\geq 2,\delta_{i-1}>0|} \\ &= c(\epsilon^2 t)^{-\xi M}(\epsilon^{\frac{1}{2}}\triangle^{-\frac{1}{2}})^{2|\delta_i=1|}(\epsilon\triangle^{-\frac{1}{2}})^{|\delta_i=0|}[\epsilon^{\frac{1}{K}}]^{K|\delta_i=2,p_i\geq 2,\delta_{i-1}>0|} \\ &\leq c(\epsilon^2 t)^{-\xi M}\max\left(\epsilon^{\frac{1}{2}}\triangle^{-\frac{1}{2}}, \epsilon\triangle^{-\frac{1}{2}}, \epsilon^{\frac{1}{K}}\right)^n. \end{aligned} \tag{4.5.32}$$

For the *choice* of $a = \frac{K}{K+1}$, one can check that the dominant term is $\epsilon^{\frac{1}{2}}\triangle^{-\frac{1}{2}}$, and we have the bound $c(\epsilon^2 t)^{-\xi M}\epsilon^{\frac{n}{2(K+1)}}$. Provided $c^* < \frac{1}{4(K+1)}$, this bound is compatible with (4.5.5).

The main problem with considering $\theta > 1$ is the inequality (4.5.30). When $m(\pi) = M$, after bounding the terms we still have to choose M, thus we can treat the terms involving θ separatedly, and in the end choose a number of iterations M such that the bound is as expected. This is possible because we can relate $|\delta_i = 2|$ with both M and n. When $m < M$, however, we lose a "variable" and we have inequalities only relating n and the other terms, $|\delta_i = 0, 1|$. While we could then group all the terms, and in the end have a bound for w'_π where the exponent is of the order of θ, whenever we try to group the terms, in the end the parameter θ ends up having no real effect in our bounds. For the case below, the argument is slightly different from the above, but it is also based on (4.5.30), which leads again for our bounds to fail. Of course, the same holds for the bounds for $w''_{\pi,H}(\underline{x},t)$. In terms of computations, we found the problem of getting a bound for w_π of order $\epsilon^{\theta n}$ similar to the one for $\theta < 1$. The idea to treat the exponents when $\theta < 1$ was to group them in such a way that, if $m = M$, we choose M accordingly, or if $m < M$, derive appropriate inequalities. While for $m < M$ we can further restrict the interval for t such that $\epsilon^{-\theta}t \geq 1$ (which is, clearly, not optimal), when $m = M$ we have the same issue: while for $\theta > 1$ we lose the order of the exponent, for $\theta < 1$ we have to make more restrictions, and in the end our bounds are not useful. We know from the original article, [24], that this method does not work for $\theta = 0$. Still, one might expect that works for some non trivial interval in $(0, 1)$. Unfortunately, to our knowledge, this is not the case. For $\theta > 1$, our conjecture is very reasonable, and since this method works for $m = M$, there might sill be something missing. Now that we exposed, in specific, the issue with both $\theta < 1$ an $\theta > 1$, for the following cases we will bound $\epsilon^{-S_4^\theta(m)}$ by 1, and conclude the exposition of [24].

For each case, the arguments for bounding $w''_\pi(\underline{x},t)$ are very similar to the ones used to bound $w''_\pi(\underline{x},t)$. We "play" with the exponents and use analogous bounds for the terms in *the last cluser*. The remaining terms are bounded similarly. We recall that from Proposition 4.4.5 we have

$$\begin{aligned} w''_{\pi,H}(\underline{x},t) &\left\{\leq c(\epsilon^2 t)^{-M\xi}\triangle^{-S_1(H-1)}\epsilon^{S_2(H-1)}t^{S_3(H-1)}\epsilon^{-S_4^{(\theta)}(H-1)}\right\}\times \\ &\times\left\{c(\epsilon^2\triangle)^{-M\xi}(\epsilon^{-2}\triangle)^{-\frac{1}{4}|G_H|}\triangle^{\frac{1}{2}\left|\delta_{i\geq H}=2\right|}\epsilon^{(\theta-1)\left|\delta_{i\geq H}=2\right|}\right\}, \end{aligned} \tag{4.5.33}$$

with the $S_k(\cdot)$ exponents as in Proposition 4.4.1

- $m(\pi) = M, w''_{\pi,H}, \triangle = \epsilon^a, \theta \geq 1.$

Let $|\delta_i = j|_{<H} : |i < H : \delta_i = j|$ and $|\delta_i = j|_{\geq H} := |i \geq H : \delta_i = j|$ (the definition is analogous for any other set). If, for example, we have $H \geq \frac{M}{2}$, then we can bound the second factor in (4.4.5) to get

$$w''_{\pi,H}(\underline{x},t) \leq c(\epsilon^{-2}t)^{-\xi M}\epsilon^{b(H-1)} \leq c(\epsilon^{-2}t)^{-\xi M}\epsilon^{\frac{b}{2}(M-2)}, \tag{4.5.34}$$

where we proceeded exactly as for the analogous case for w'_π, thus, b is the same. Choosing M such that $\frac{1}{2}b(M-2) > 2n$ we get $w''_{\pi,H}(\underline{x},t) \leq c(\epsilon^{-2}t)^{-\xi M}\epsilon^n$, where we need only that $c^* < 1/2$.

If $H < \frac{M}{2}$, we bound the first factor in (4.5.33) by 1 to get

$$w''_{\pi,H}(\underline{x},t) \leq c(\epsilon^{-2}t)^{-\xi M}(\epsilon^2\triangle)^{-\frac{1}{4}|G_H|}\triangle^{\frac{1}{2}|\delta_i=2|_{\geq H}} = [(\epsilon^2\triangle)^{-\frac{1}{8}}]^{2|G_H|}[\triangle^{\frac{1}{2}}]^{|\delta_i=2|_{\geq H}}. \tag{4.5.35}$$

Since

$$\begin{aligned} &-2G_H + |\delta_i = 0, 1, \{k_i, l_i\} \cap A_{H-1} \neq \emptyset|_{\geq H} \leq 0, \\ &-|\delta_i = 0, 1, \{k_i, l_i\} \cap A_{H-1} = \emptyset| + K|\delta_i = 2|_{\geq H} \geq 0, \end{aligned} \tag{4.5.36}$$

we have that

$$w''_{\pi,H}(\underline{x},t) \leq c(\epsilon^{-2}t)^{-\xi M}\max\left([(\epsilon^2\triangle)^{-\frac{1}{8}}], \triangle^{\frac{1}{2}}\right)^{2|G_H|+|\delta_i=2|_{\geq H}}. \tag{4.5.37}$$

Denoting the maximum by $\epsilon^{b'}$, and applying the bounds in (4.5.36), we have

$$w''_{\pi,H}(\underline{x},t) \leq c(\epsilon^{-2}t)^{-\xi M}\epsilon^{b'(|\delta_i=0,1,\{k_i,l_i\}\cap A_{H-1}\neq\emptyset|_{\geq H}+|\delta_i=2|_{\geq H})}. \tag{4.5.38}$$

Now assume that $|\delta_i = 0,1,\{k_i,l_i\}\cap A_{H-1}\neq\emptyset|_{\geq H} \geq \frac{1}{2}|\delta_i=0,1|_{\geq H}$. Then, from (4.5.36) we get that $|\delta_i=2|_{\geq H} \geq \frac{1}{2K}|\delta_i=0,1|_{\geq H}$, and

$$\begin{aligned}&|\delta_i = 0,1,\{k_i,l_i\}\cap A_{H-1}\neq\emptyset|_{\geq H} + |\delta_i=2|_{\geq H} \geq \frac{1}{2}|\delta_i=2|_{\geq H} + \frac{1}{4K}|\delta_i=0,1|_{\geq H}\\ &\geq \frac{M-H}{4K} \geq \frac{M}{8K},\end{aligned} \tag{4.5.39}$$

where we used that $K>1$, and this case is done. Now say that we have $|\delta_i = 0,1,\{k_i,l_i\}\cap A_{H-1}\neq\emptyset|_{\geq H} < \frac{1}{2}|\delta_i=0,1|_{\geq H}$. Then we can simply bound as follows

$$\epsilon^{b'(|\delta_i=0,1,\{k_i,l_i\}\cap A_{H-1}\neq\emptyset|_{\geq H}+|\delta_i=2|_{\geq H})} \leq \epsilon^{\frac{1}{2}b'(M-H)}. \tag{4.5.40}$$

Taking M such that $M-H\geq M/2$ we are done.

- $m(\pi)=M, w''_\pi, \triangle = \frac{t}{M+1}, \theta\geq 1$

For the first factor in (4.5.33) we bound as in w'_π. For the second, we bound everything by $\triangle^{|\delta_i=2|_{\geq H}}$ to get

$$w''_{\pi,H}(\underline{x},t) \leq c\epsilon^{-4\xi M}\epsilon^{\frac{1}{2}\beta^*|\delta_i=2|_{<H}}\triangle^{|\delta_i=2|_{\geq H}}. \tag{4.5.41}$$

Now that the exponents are only functions of $\delta_i=2$, one can use the same argument as in the analogous case for w'_π . In the end, we will need $c^*<1/2$. We remark that these arguments are exactly the ones we want to use to show the specific bound already mentioned, for $\theta>1$, since $S_4^{(\theta)}(m)$ is function of essentially $\delta_i=2$, and also works for $\triangle=\epsilon^a$. As already mentioned, the problem lies in the case $m<M$.

- $m(\pi)<M, w''_\pi, \triangle=\epsilon^a, \theta\geq 1.$

Since all particles die before the iteration M, we derive the analogous of (4.5.28). Recalling the definiton of G_H in (4.4.27), since for every particle $l\in A_{H-1}\ \exists i\geq H : \delta_i=0,1, l\in\{k_i,l_i\}$, it is not difficult to see that we have

$$|A_{H-1}| \leq 2|G_H| + K|\delta_i=2|. \tag{4.5.42}$$

The terms in the first factor can be bounded as in the analogous case for w'_π. Thus, the bound arising from these terms is $\epsilon^{\frac{1}{2(K+1)}(n)} \leq \epsilon^{\frac{1}{2(K+1)}(n-|A_{H-1}|)}$, modulo the $(\epsilon^2 t)^{-\xi M}$ term. We used the last bound in order to group it with the one from the last cluster, as we will see. The terms from C_H, the second factor, can be bounded as follows

$$(\epsilon^2\triangle)^{-\frac{1}{4}|G_H|}\triangle^{|\delta_i=2|_H} = (\epsilon^2\triangle)^{-\frac{1}{8}(2|G_H|)}\triangle^{\frac{1}{K}(K|\delta_i=2|_H)} \leq \max\left((\epsilon^2\triangle)^{-\frac{1}{8}}\triangle^{\frac{1}{2K}}\right)^{|A_{H-1}|}. \tag{4.5.43}$$

If $|A_{H-1}|\geq n$, we are done. Otherwise, the largest term in the previous display is $\triangle^{\frac{1}{2K}}$, by the choice of a. In the end, we will need $c^* < \frac{1}{4(K+1)}$.

- $m(\pi) < M, w''_\pi, \triangle = \triangle = \frac{t}{M+1}, \theta \geq 1$

By the same arguments as for the previous case, we bound the second factor by

$$(\epsilon^2 \triangle)^{-\xi M} \max\left((\epsilon^2 t)^{-\frac{1}{8}} t^{\frac{1}{2K}}\right)^{|A_{H-1}|}. \tag{4.5.44}$$

Again, either if $|A_{H-1}| < n$ or $|A_{H-1}| \geq n$, we bound the first factor in (4.5.33) as in the analogous case for w'_π, to arrive at the bound $(\epsilon^2 t)^{-\xi M}(\epsilon t^{-\frac{1}{2}})^{\frac{n}{1+K}} \leq (\epsilon^2 t)^{-\xi M}(\epsilon^{-2} t)^{-\frac{n-|A_{H-1}|}{2(1+K)}}$. If $|A_{H-1}| \geq n$ then we are done. Otherwise, recalling that $\epsilon^{-2} t \geq 1$, and that $t \geq \epsilon^{\beta^*}$, we have

$$\max\left((\epsilon^2 t)^{-\frac{1}{8}} t^{\frac{1}{2K}}\right) \leq \begin{cases} t^{\frac{1}{2K}}, & t \geq \epsilon^{\frac{2K}{K+4}} \\ (\epsilon^{-2} t)^{-\frac{1}{8}}, & \epsilon^2 \leq t \leq \epsilon^{\frac{2k}{K+4}} \end{cases}. \tag{4.5.45}$$

To uniformize the bounds, note that $t^{\frac{1}{2K}} \leq (\epsilon^{-2} t^{-1})^{\frac{1}{2(K+2)}}$ for any $\epsilon^{\frac{2K}{K+4}} \leq t \leq \epsilon^a$. When the dominant term is $(\epsilon^{-2} t)^{-\frac{1}{8}}$, clearly $(\epsilon^{-2} t)^{-\frac{1}{8}} \leq (\epsilon^{-2} t)^{-\frac{1}{2(1+K)}}$. In the end, we need $c^* < \frac{1}{2(2+K)}$.

Chapter 5

Matrix Product Ansatz

In this chapter we explore the *Matrix Product Ansatz* (MPA) for systems with the SSEP dynamics acting in the bulk, and general boundary dynamics acting in a window of size 2. For some models, the MPA allows us to obtain the probability of any configuration given through a matrix product. For the model studied through this thesis, we know that the *Bernoulli product measure* is not an invariant measure, and we have no information regarding what the invariant measure, in general, might be. The reason that the Bernoulli product measures with parameter γ, ν_γ^N, are not invariant for the whole choice of the parameters α_i, γ_i, β_i and δ_i, is the following. Suppose that there exists a constant γ such that ν_γ^N is invariant. From (2.0.39) for any cylindrical function f we must have

$$\nu_\gamma^N(\mathcal{L}f) = 0. \tag{5.0.1}$$

Choosing $f(\eta) = \eta(1)$, one can we see that for (5.0.1) to be true, we need to impose $\gamma = \frac{\alpha_1}{\alpha_1+\gamma_1}$. For the choice $f(\eta) = \eta(2)$ we see that we need to impose $\alpha_2 = \gamma_2$. For the choice $f(\eta) = \eta(1)\eta(2)$, we see that we need to impose $\gamma_1\alpha_2 = 0$. Then, if $\alpha_2 = 0$, we also have $\gamma_2 = 0$ and we are back to the linear SSEP for which one can prove that the Bernoulli product measures ν_γ^N are, in fact, reversible. If we assume that this is not the case, then we need to have $\gamma_1 = 0$. For the choice $f(\eta) = \eta(1)(1 - \eta(2))$ we see that we get a contradiction. So, apart the case of the linear SSEP, these measures are not invariant.

As mentioned in the first chapter, we know that for $K = 2$, $\alpha_2 = \gamma_2$ and $\beta_2 = \delta_2$ there are no explicit correlations in the integral formulation obtained through Dynkin's martingale. Under this choice of parameters, one can simply compute Kolmogorov's equation and, under the stationary regime, explicitly solve the system. For a more general choice of parameters that is not possible due to the presence of correlations. In [26] it was shown that for general K, but $\alpha_i = 0 = \delta_i, \gamma_i = 1 = \beta_i$ and $\theta = 1$ the (stationary) empirical profile is associated to a linear function in the continuum setting where the boundary conditions are solution of a polynomial of degree K. In this way, the MPA formulation is a possible candidate to obtain some information regarding the stationary state. Unfortunately, as we will see, the current methodology is not enough to solve our problem. Thus, we present here the problems with the current methodology, study the SSEP with *linear* reservoirs to give the reader some context (Subection 5.1.3), and propose an extension for the methodology (Section 5.2). We believe that the first indicator that our extension works is the *consistency of the algebra*, which is what we will show in the Subsection 5.2.2.

5.1 Usual Methodology

The idea of the *MPA* is to assume that the probability of a configuration η in the stationary regime can be formulated as a product of matrices. These matrices, in general of infinite dimension, must satisfy a set of rules induced by the generator of the process. These rules are given by an *algebra*. From the *Kolmogorov equation* (2.0.32), one gets that the time derivative of the probability of a configuration with respect to the stationary measure vanishes, thus the stationary measure lies at the kernel of the generator. To get this condition, the current methodology relies on a cancelling mechanism (also known as *telescopic rules*), proposed in [8]. Since its proposal it has been shown that, under some conditions, these rules are consequence of the integrability of the model, and are closely related to the well known *Bethe Ansatz* [32]. Unfortunately, this formulation only seems to work for "simple" models. Here, simple does not mean the complexity of the dynamics *per se*, but the dimension of the action of each local operator that defines the dynamics. For closed boundaries, or open but with each boundary acting on one site less than the bulk dynamics [18] we have a theorem that states the existence of such matrices, if one can show that the algebra is consistent [19]. Moreover, even showing that such factorization exists, the problem of computing any physical quantity is not trivial, due to the complex structure of these matrices (for more details in this topic check [6]). Much simpler are the cases when one does not need the representation of the matrices to compute any quantity - which was our main motivation for solving this problem. Nevertheless, existence of such matrices is still a problem not studied well enough in the literature.

We analise deeply the *linear SSEP* with general rates from the algebraic point of view, in order to give the reader some insight to what we will do next. Along the way, we make some corrections to the algebra for the linear SSEP with *slow* reservoirs. To our knowledge, the subtle incapacities of the paradigmatic algebra for the linear SSEP with slow reservoirs were never detected in the current literature. Afterwards, we propose a natural extension of this method, and show under what conditions the algebra is consistent. Given the extent of this thesis, we will not compute any quantity, but only pave the ground for a future work. Before showing (or not) the existence of such matrices, we prefered to take advantage of our algebra and check the consistency, given that there is no theorem that guarantees that our method works yet. Both the representation of the matrices and estimates for quantities of interest will be addressed in a forthcoming work.

Although previous methodology still "works" for our model, the restricitions to the parameters space are too strong. We were able to relax these restrictions by considering some extra boundary matrices (or vectors, depending on the point of view). We have no knowledge of work being done in this direction in the literature. Up to now, our extension has proven to be successful.

5.1.1 Mathematical framework

In this section we will introduce the mathematical framework for the MPA. We will use the *tensor product* formalism, given its simplicity for constructing the probability vectors and in general more compact expressions. We will make a brief summary of the theory from [32]. For a more detailed exposition, and relationships with *integrable models* and the *Bethe Ansatz* we direct the reader to the aforementioned work. By the time of the writing of this thesis we noticed that a new formalism has been recently developed, in the context of *Hidden Markov Chains*. For more details, we direct the reader to [2].

Recall that for each site $i \in \Lambda_N$ (we now denote a site by i given the different context of this chapter) we have a local configuration variable $\eta_i \in \{0,1\}$, where $\eta_i = 0$ if the site is empty and $\eta_i = 1$ if the site is full. Note that we have 2^{N-1} possible configurations. In order to express the probabilities of configurations in a vector form, we need to construct a vector space with a chosen basis. Thus, to each configuration $\eta = (\eta_1, \dots, \eta_{N-1})$ we associate a basis vector $|\eta_1, \dots, \eta_{N-1}\rangle$, and to each η_i we associate a basis vector $|\eta_i\rangle$ of $\mathbb{C}$.

Definition 5.1.1. For $\eta_i = 0, 1$ the vector $|\eta\rangle$ is defined by $|1\rangle = (0,1)^\dagger, \quad |0\rangle = (1,0)^\dagger$, where $.^\dagger$ denotes the transposed vector. $\{|0\rangle, |1\rangle\}$ consititutes the canonical basis of $\mathbb{C}$.

Definition 5.1.2. The vector $|\eta_1, \dots, \eta_{N-1}\rangle$ is defined by $|\eta_1, \dots, \eta_{N-1}\rangle = |\eta_1\rangle \otimes |\eta_2\rangle \otimes \cdots \otimes |\eta_{N-1}\rangle$.

Example 5.1.3. The definition above states that we can add more sites by simply taking the tensor product of elements of our basis. In this way, while the local configuration η_i take values

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{5.1.1}$$

the basis associated to two sites takes the form:

$$|0\rangle \otimes |0\rangle = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}^\dagger, \quad |0\rangle \otimes |1\rangle = \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix}^\dagger, \quad |1\rangle \otimes |0\rangle = \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix}^\dagger, \quad |1\rangle \otimes |1\rangle = \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix}^\dagger, \tag{5.1.2}$$

where each element corresponds to the *empty lattice*, *second site full*, *first site full* and *full lattice*, respectively.

Now we can define the probability vector as

$$\begin{aligned} |P_t\rangle &= (P_t(0,\dots,0,0), P_t(0,\dots,0,1), \dots, P_t(1,\dots,1,1))^\dagger \\ &= \sum_{0 \le \eta_1,\dots,\eta_{N-1} \le N} P_t(\eta_1,\dots,\eta_{N-1}) \, |\eta_1\rangle \otimes \cdots \otimes |\eta_{N-1}\rangle, \end{aligned} \tag{5.1.3}$$

where to each configuration the associated probability is stored as a coefficient. We are interested only on models such that the dynamics can be encoded by a Markov matrix (a real square matrix with each row summing to 1) that can be decomposed as a sum of local operators acting only on two neighbor sites. For closed boundaries this dynamics can be encoded by the operator

$$M = \sum_{i=1}^{N-1} m_{i,i+1}, \tag{5.1.4}$$

with $m_{i,i+1}$ a local jump operator acting on sites $i, i+1$: $m_{i,i+1} = 1^{\otimes i-1} \otimes m \otimes 1^{\otimes N-i-1}$, with the conventions $m^{\otimes 0} = 1$ and $m^{\otimes 1} = m$; where m is a matrix of size 4×4 acting on the vector space $\mathbb{C} \otimes \mathbb{C}$ (*i.e.,* two adjacent sites), and 1 is the identity matrix with dimension 2×2.

Remark 5.1.4. Notice that under this notation, for local configurations (ξ, ξ') and (τ, τ') with $(\xi, \xi') \neq (\tau, \tau')$, the element $\langle \xi | \otimes \langle \xi' | \, m \, | \tau \rangle \otimes | \tau' \rangle$ corresponds to the probability rate that the system goes from the configuration $(\eta_1, \dots, \eta_{i-1}, \tau, \tau', \eta_{i+2}, \dots, \eta_{N-1})$ to $(\eta_1, \dots, \eta_{i-1}, \xi, \xi', \eta_{i+2}, \dots, \eta_{N-1})$. The reader can thus make a clear correspondence between m and the transition rates in the definition of the generator in (2.0.27). Moreover, this rate depends on the *local* configuration only, and not on the states of the other sites. This is important because the rules induced by our dynamics are thus *local*, taking into consideration only the sites they act on. Altough this makes the MPA problem simpler, as we will see we also lose some important dependencies in our system.

Example 5.1.5. For the *SSEP* dynamics, with respect to the *ordered* vector basis $\{|0\rangle \otimes |0\rangle, |0\rangle \otimes |1\rangle, |1\rangle \otimes |0\rangle, |1\rangle \otimes |1\rangle\}$, we have

$$m = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{5.1.5}$$

In practice, to obtain this matrix, one fixes a base, then the entry $m(i,j)$ corresponds to the transition rate to go from the local configuration associated to the basis vector i to the local configuration associated to the basis vector j. This will be more clear when we relate more explicitly this local operator with the generator.

In the case of *open boundaries* with coupled reservoirs acting on a single site each, the operator M simply takes the form:

$$M = \sum_{i=1}^{N-1} m_{i,i+1} + B_L + B_R, \tag{5.1.6}$$

where $B_L = b_L \otimes 1^{\otimes(N-2)}$ and $B_R = 1^{\otimes(N-2)} \otimes b_R$, and b_L, b_R are 2×2 matrices acting on the first (resp. last) site.

Example 5.1.6 (Continuation)**.** Consider the dynamics where, at the left, a particle can be injected to the first site with rate α if that site is empty, and removed with rate γ if the site is full (the classical *SSEP with "linear" reservoirs*, the particular case of the dynamics studied through this work with $K = 1$). For the right, we exchange α to β and γ to δ. Under the *ordered* basis $\{|0\rangle, |1\rangle\}$, these jump operators are written as

$$b_L = \begin{pmatrix} -\alpha & \gamma \\ \alpha & -\gamma \end{pmatrix}, \qquad b_R = \begin{pmatrix} -\beta & \delta \\ \beta & -\delta \end{pmatrix}. \tag{5.1.7}$$

5.1.2 General idea

As already stated, the main idea for the *MPA* is to assume that the probability of a configuration can be factorized into a matrix product. For that, we associate a matrix E to an empty site, and a matrix D to a full site. In this way, we can associate a configuration η to the *ordered* product

$$\prod_{i=1}^{N-1} \left[(1-\eta_i)E + \eta_i D\right]. \tag{5.1.8}$$

In order to get a real number, we apply a *trace operator* (since we will only work with 1 specie of particles (*i.e.,* the particles are not distinguished) one can simply state vectors $\langle W|$ and $|V\rangle$) to the product on the previous display, and normalize it to send the result to $[0, 1]$. Thus, since under the *invariance condition* (2.0.39) we have

$$(\mathcal{L}^\dagger \mu_{ss}^N)(\eta) = 0, \quad \forall \eta \in \Omega_N, \tag{5.1.9}$$

where we recall that $(\mu\mathcal{L})(\eta) \equiv (\mathcal{L}^\dagger\mu)(\eta)$ (the transposed operator $\mathcal{L}^\dagger$ is the adjoined of $\mathcal{L}$ with respect to the standard scallar product), one can express physical quantities with respect to the stationary measure as

$$\mu_{ss}^N(\eta) = \frac{1}{Z_{N-1}} \langle W| \prod_{i=1}^{N-1} [(1-\eta_i)E + \eta_i D] |V\rangle, \tag{5.1.10}$$

where Z_{N-1} is the normalization for a system with $N-1$ sites. Letting $C := D + E$, it is easy to see that the normalization Z_{N-1} takes the simple form $Z_{N-1} = \langle W| C^{N-1} |V\rangle$. Recalling the notation in (5.1.3), thanks to the tensor product formalism one can recast (5.1.3), with a clear abuse of notation, as

$$|P\rangle = \frac{1}{Z_{N-1}} \langle W| \begin{pmatrix} E \\ D \end{pmatrix}^{\otimes(N-1)} |V\rangle. \tag{5.1.11}$$

Recalling the *master equation* (2.0.32), one can check that, with this notation, the mentioned equation can be written as $\partial_t |P_t\rangle = M |P_t\rangle$, where M is the operator defined in (5.1.6). Under the stationary state, the left hand-side of the *master equation* vanishes and we get the invariance condition (5.1.9). Defining the *Intensity matrix* H by the matrix elements

$$H_{\xi,\eta} := \begin{cases} -c(\xi,\eta) & \xi \neq \eta \\ \sum_{\eta\in\Omega_N\setminus\xi} c(\eta,\xi) & \xi = \eta \end{cases} \tag{5.1.12}$$

where $c(\xi, \eta)$ are the transition rates, as defined in (2.0.26), one can check that $M = -H$ is the intensity matrix of the process generated by $\mathcal{L}$, and we have $(\mathcal{L}f)(\eta) = -\sum_{\xi\in\Omega_N} H_{\xi,\eta} f(\xi)$.

As already mentioned, the current methodology relies on forcing condition (5.1.9) through a *telescopic rule*, which we state on the following definition.

Definition 5.1.7. Let $\mathcal{X} := (X_1, X_2)^\dagger$ and $X := (E, D)^\dagger$. Given a bulk local jump operator m, we say that X and $\mathcal{X}$ satisfy the *bulk telescopic relation* if

$$mX \otimes X = X \otimes \mathcal{X} - \mathcal{X} \otimes X. \tag{5.1.13}$$

Moreover, we call the vector $\mathcal{X}$ the *auxiliary vector*.

Proposition 5.1.8. *If the vectors X, $\mathcal{X}$ satisfy the bulk telescopic relation, and we also have*

$$\langle W| b_L X = \langle W| \mathcal{X} \qquad b_R X |V\rangle = -\mathcal{X} |V\rangle \tag{5.1.14}$$

then the matrix product state (5.1.11) satisfies $M|P\rangle = 0$. If $|P\rangle \neq 0$ this formulation provides the stationary state associated to M.

Proof. The proof of this result is simple and relies only on computing $M\,|P\rangle$ and applying the telescopic rules. For details check [32]. $\square$

Although the telescopic rules guarantee that we have the stationary state (if one has no inconsistencies on the algebra and the normalization does not degenerate), there is no guarantee to whether one can easily compute any quantity. We know that some models are *algebraically solvable*, *i.e.,* one can compute any quantity using only the induced algebra. Nevertheless, to guarantee that matrices that satisfy these rules exist, one has to either find them, or be in the conditions of Definition 5.1.7 and Proposition 5.1.8. Advances on the generalization of Poposition 5.1.8 have been very few. For any multispecies dynamics it is known that Proposition 5.1.8 holds with only a slight modification on the basis and, clearly, the jump operator M. For more details regarding multispecies dynamics see [32] and [29]. Still, one has to restrict to the action of the boundaries on 1 site only each, and the bulk on 2 sites. A natural and simple generalization was proved successfull in [18], where one can extend the action of each reservoir to $r-1$ sites - still, the bulk must act on r sites. The reason for this is simply that the cancelation mechanism in Definition 5.1.7 still holds with this choice. Only very recently (in fact, we noticed this by the time of the writing of this thesis) that the *open zero-range process* was shown to be exactly solvable through a *MPA* [5] - both algebraically and through the matrices representation. On the next subsection we will analyse the algebraic formulation for the *linear SSEP*. Since this example will be lenghty enough, we will give its own subsection. The *MPA* formulation was first introduced in [8] in a more "heuristic" direction to express correlations for the TASEP, and since then the MPA has been shown to be a *reformulation* of a problem, for systems with reservoirs acting in one site each, and the bulk dynamics acting in two [19], thus not an *ansatz* method under these conditions. The slow boundary case was first studied in [28]. Here we introduce a new look on the *linear SSEP*, presenting a slight correction in the algebra for the slow boundary case, with a completely algebraic approach and trying to "dismistify" the choice for the usual algebra.

5.1.3 A new look on the linear SSEP

Recalling the jump matrices (5.1.5) and (5.1.7) one can compute the telescopic rules in Definition 5.1.7:

$$m\begin{pmatrix}E\\D\end{pmatrix}^{\otimes 2} = \begin{pmatrix}E\\D\end{pmatrix}\otimes\begin{pmatrix}X_1\\X_2\end{pmatrix} - \begin{pmatrix}X_1\\X_2\end{pmatrix}\otimes\begin{pmatrix}E\\D\end{pmatrix} \tag{5.1.15}$$

to get

$$\begin{aligned} 0 &= EX_1 - X1E,\\ ED - DE &= EX_2 - X_1D,\\ -ED + DE &= DX_1 - X_2E,\\ 0 &= DX_2 - X_2D, \end{aligned} \tag{5.1.16}$$

and for the boundaries

$$\langle W| b_L \begin{pmatrix} E \\ D \end{pmatrix} = \langle W| \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \Leftrightarrow \begin{pmatrix} \langle W| - \alpha E + \gamma D \\ \langle W| \alpha E - \gamma D \end{pmatrix} = \begin{pmatrix} \langle W| X_1 \\ \langle W| X_2 \end{pmatrix}, \tag{5.1.17}$$

$$b_R \begin{pmatrix} E \\ D \end{pmatrix} |V\rangle = - \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} |V\rangle \Leftrightarrow \begin{pmatrix} -\beta E + \delta D |V\rangle \\ \beta E - \delta D |V\rangle \end{pmatrix} = \begin{pmatrix} -X_1 |V\rangle \\ -X_2 |V\rangle \end{pmatrix}. \tag{5.1.18}$$

Instead of choosing *a priori* the auxiliary vector in (5.1.7) as the simplest possible that works, as in [6], we will exploit the relations to see which choice is the most natural. Bulk relations (5.1.16) can be written as

$$[E, X_1] = 0, \quad [E, D] = EX_2 - X_1 D, \quad [E, D] = X_2 E - DX_1, \quad [D, X_2] = 0, \tag{5.1.19}$$

where $[E, D]$ is the commutator of E and $D : [E, D] = ED - DE$. Thus we must have the consistency condition

$$[E, D] = EX_2 - X_1 D = X_2 E - DX_1 \Leftrightarrow [E, X_2] = -[D, X_1]. \tag{5.1.20}$$

Recalling that $C := D + E$, summing $[D, X_2] = 0$ and $[E, X_1] = 0$ on both sides on the previous display we have by definition

$$[C, X_2] = -[C, X_1] \Leftrightarrow [C, X_1 + X_2] = 0. \tag{5.1.21}$$

Now let us look at the boundary rules (5.1.17). In order for this algebra to be consistent we must have

$$((\langle W| - \alpha E + \gamma D = \langle W| X_1 = - \langle W| X_2) \Rightarrow \langle W| (X_1 + X_2) = 0. \tag{5.1.22}$$

And for the right we also have $(X_1 + X_2) |V\rangle = 0$. Unfortunately, relying only on these relations for the auxiliary vector we cannot compute any quantity. Before proceeding, will define more precisely what we mean by algebraically solvabe.

Definition 5.1.9. Given a *MPA* formulation for a dynamics where each reservoir acts on K sites (that is, b_L, b_R acts on $\mathbb{C}^K$), we say that the *bulk* is *algebraically solvable* if we can express configurations with a fixed number of particles in the bulk as functions of configurations with particles at a distance $\leq K$ from a boundary.

We say the *boundary* is *algebraically solvable* if we can express local configurations with a fixed number of particles in a distante $\leq K$ from a boundary as a function of the normalization constant only.

The bulk and boundary rules suggest that we have $X_2 = -X_1$. As we will see, this is enough to compute *at least* up to two-sites correlations. Note that the empirical mean at a site x in a system of $N - 1$ particles has the formulation

$$\mathbb{E}_{\mu_{ss}^N}[\eta(s)] := < \eta(x) >_{N-1} = \frac{1}{Z_{N-1}} \langle W| C^{x-1} D C^{N-1-x} |V\rangle, \tag{5.1.23}$$

where μ_{ss}^N is the stationary measure. Now note also that $[D, C] = [D, E]$, since $[D, D] = 0$ by definition.

Moreover, with the choice $X = X_2 = -X_1$ our rules take the form

$$[E, X] = [D, X] = 0, \quad [D, E] = XC, \quad \langle W| \alpha E - \gamma D = \langle W| X, \quad \delta D - \beta E |V\rangle = X |V\rangle . \tag{5.1.24}$$

In this way, since $[D, C] = XC$, one can show by induction that

$$C^{x-1}D = C^{x-2}DC - XC^{x-1} = \cdots = DC^{x-1} - (x-1)XC^{x-1}. \tag{5.1.25}$$

Thus, the auxiliary vector defined by $\mathcal{X} = (-X, X)^\dagger$ induces the algebraic solvability of the bulk. Using this, the empirical mean takes the form

$$< \eta(x) >_{N-1} = \frac{1}{Z_{N-1}} \langle W| DC^{N-2} |V\rangle - (x-1)\frac{1}{Z_{N-1}} \langle W| X_1 C^{N-2} |V\rangle . \tag{5.1.26}$$

Noticing that

$$\langle W| X = \langle W| \alpha E - \gamma D = \langle W| - (\alpha+\gamma)D + \alpha C \Leftrightarrow \langle W| D = \langle W| (\alpha+\gamma)^{-1}(-X + \alpha C), \tag{5.1.27}$$

one concludes that

$$< \eta(x) >_{N-1} = \frac{\alpha}{\alpha+\gamma} - \frac{\langle W| XC^{N-2} |V\rangle}{Z_{N-1}} \left((x-1) + \frac{1}{\alpha+\gamma} \right). \tag{5.1.28}$$

Again, $X_2 = -X_1$ induces the algebraic solvability of the left boundary, and by symmetry also the right boundary. We conclude that the *linear SSEP* is algebraically solvable under this choice. Note that we do not need to make any specification regarding the nature of X. This can either be a matrix or a constant. Of course, if we send "particle" D to the right in (5.1.26) instead, we must have the same result. In this way, we also have

$$< \eta(x) >_{N-1} = \frac{1}{Z_{N-1}} \langle W| C^{N-2}D |V\rangle + (N-1-x)\frac{\langle W| C^{N-2}X |V\rangle}{Z_{N-1}}. \tag{5.1.29}$$

By analogous computations as in (5.1.27) we have that $D |V\rangle = (\beta+\delta)^{-1}(\beta C + X) |V\rangle$. Thus,

$$< \eta(x) >_{N-1} = \frac{\beta}{\beta+\delta} + \frac{\langle W| C^{N-2}X |V\rangle}{Z_{N-1}} \left(N-2-(x-1) + \frac{1}{\beta+\delta} \right). \tag{5.1.30}$$

In this way, to have consistency on the computation for the empirical mean we must equate (5.1.28) and (5.1.30):

$$\begin{aligned} &\frac{\alpha}{\alpha+\gamma} - \frac{\langle W| XC^{N-2} |V\rangle}{Z_{N-1}} \frac{1}{\alpha+\gamma} = \frac{\beta}{\beta+\delta} + \frac{\langle W| C^{N-2}X |V\rangle}{Z_{N-1}} \left(N-2 + \frac{1}{\beta+\delta} \right) \\ \Leftrightarrow &\frac{\langle W| X_1 C^{N-2} |W\rangle}{Z_{N-1}} = \frac{\frac{\alpha}{\alpha+\gamma} - \frac{\beta}{\beta+\delta}}{N-2+\frac{1}{\alpha+\gamma}+\frac{1}{\beta+\delta}}. \end{aligned} \tag{5.1.31}$$

Replacing this in (say) (5.1.28) one gets

$$< \eta(x) >_{N-1} = \frac{\alpha}{\alpha+\gamma} - \left(x-1+\frac{1}{\alpha+\gamma} \right) \frac{\frac{\alpha}{\alpha+\gamma} - \frac{\beta}{\beta+\delta}}{N-2+\frac{1}{\alpha+\gamma}+\frac{1}{\beta+\delta}}. \tag{5.1.32}$$

Depending on the structure of the matrices D, E, each component of the auxiliary vector $X = (-X, X)^\dagger$ does not need to be a constant. Still, when computing higher correlations the constant choice seems more obvious. But before that, note that since we chose general rates, one can also study the *slow boundary* SSEP easily from the previous computations. Let

$$\alpha = N^{-\theta}\alpha', \quad \beta = N^{-\theta}\beta', \quad \gamma = N^{-\theta}\gamma', \quad \delta = N^{-\theta}\delta', \tag{5.1.33}$$

then one gets

$$< \eta(x) >_{N-1} = \frac{\alpha}{\alpha+\gamma} - \left(x - 1 + \frac{N^\theta}{\alpha+\gamma}\right) \frac{\frac{\alpha}{\alpha+\gamma} - \frac{\beta}{\beta+\delta}}{N - 2 + \frac{N^\theta}{\alpha+\gamma} + \frac{N^\theta}{\beta+\delta}}. \tag{5.1.34}$$

We will denote $N^{-\theta}\alpha'$ by $N^{-\theta}\alpha$, $N^{-\theta}\beta'$ by $N^{-\theta}\beta$, and so on for simplicity, since it is clear that we are working on the slow boundary case. Clearly, the empirical mean converges to different expressions depending on the values of θ. Letting

$$J_{N-1}(\theta) := N \frac{\langle W | \, X_1 C^{N-2} \, | V \rangle}{Z_{N-1}} = N^{1-\theta} \frac{\alpha(\alpha+\gamma)^{-1} - \beta(\beta+\delta)^{-1}}{N^{1-\theta} - 2N^{-\theta} + (\alpha+\gamma)^{-1} + (\beta+\delta)^{-1}}. \tag{5.1.35}$$

With this quantity, one can rewrite the empirical mean (5.1.34) as

$$< \eta(x) >_{N-1} \equiv \rho^N(u) = \frac{\alpha}{\alpha+\gamma} - uJ_{N-1}(\theta) + N^{\theta-1}\frac{J_{N-1}(\theta)}{\alpha+\gamma} + \frac{J_{N-1}(\theta)}{N} \tag{5.1.36}$$

where we used the substitution $u = x/(N-1)$. Computing the limit $N \to \infty$ the interested reader can check that the density $\rho(u) := \lim_{N\to\infty} \rho^N(u)$ depends on the choice of θ, as expected:

$$\rho(u) = \begin{cases} \frac{\alpha}{\alpha+\gamma} - u\frac{\alpha\delta-\beta\gamma}{(\beta+\delta)(\alpha+\gamma)}, & \theta \in [0,1), \\ \frac{\alpha(1+\beta+\delta)+\beta}{\beta+\delta+\alpha(1+\delta+\beta)+\gamma(1+\beta+\delta)} - u\frac{\alpha\delta-\beta\gamma}{\beta+\delta+\alpha(1+\delta+\beta)+\gamma(1+\beta+\delta)}, & \theta = 1, \\ \frac{\alpha+\beta}{\alpha+\beta+\delta+\gamma}, & \theta > 1. \end{cases} \tag{5.1.37}$$

In particular, the quantity $N^{-1}J_{N-1}(\theta)$ is the *discrete current* in the bulk, which is independent of the location (recall that $[D, E] = XC$). One can check that these quantities agree with [28] under the choice $\alpha + \gamma = 1$ and $\beta + \delta = 1$. As already mentioned, when computing the correlations some problems arise. The expression for the two-sites correlation ($x < y$) is given by

$$< \eta(x)\eta(y) >_{N-1} = \frac{1}{Z_{N-1}} \langle W | \, C^{x-1} D C^{y-x-1} D C^{N-1-y} \, | V \rangle . \tag{5.1.38}$$

For simplicity, denote by $< \eta >^w_{N-1}$ the weight of the configuration η, *i.e.*, $< \eta >^w_{N-1} := Z_{N-1} < \eta >_{N-1}$. Moreover, let us introduce the L/R "operators".

Definition 5.1.10. Given a configuration with a particle at site y, we let $L_y(x) < \eta >_{N-1}$ mean that we will send the "particle" y to the site $x < y$ whenever the configuration at sites $\{x, \dots, y-1\}$ is *free*, that is, there are no particles fixed in between x and y, *i.e.,* $\{x, \dots, y-1\}$. Similarly, we write $R_x(y) < \eta >_{N-1}$ when we send a particle from the position x to y with $x < y$ and no particles in $\{x+1, \ldots, y\}$. Moreover, we will write $L_y(1) \equiv L_y$ and $R_y(N-1) \equiv R_y$.

In this way, to check the consistency for the computations of the empirical mean in (5.1.31) we actually showed that Z_{N-1} is such that $(L_x - R_x) < \eta(x) >^w_{N-1} = 0$. To compute the weight for the correlation the idea is completely analogous to computing the mean: we want to send a "particle" D to the boundary, use our boundary relations and check whether we can compute all the terms explicitly. Thus,

$$L_x < \eta(x)\eta(y) >^w_{N-1} = -(x-1) \langle W| XC^{y-2}DC^{N-y-1} |V\rangle + < \eta(1)\eta(y) >^w_{N-1} . \tag{5.1.39}$$

Now we compute

$$\begin{aligned} L_y \langle W| XC^{y-2}DC^{N-y-1} |V\rangle &= \frac{\alpha}{\alpha+\gamma} \langle W| XC^{N-2} |V\rangle - (y-2+\frac{1}{\alpha+\gamma}) \langle W| XXC^{N-3} |V\rangle \\ R_y \langle W| XC^{y-2}DC^{N-y-1} |V\rangle &= \frac{\beta}{\beta+\delta} \langle W| XC^{N-2} |V\rangle + (N-y-1+\frac{1}{\beta+\delta}) \langle W| XXC^{N-3} |V\rangle . \end{aligned} \tag{5.1.40}$$

Thus the condition $(L_y - R_y) \langle W| XC^{y-2}DC^{N-y-1} |V\rangle = 0$ reads

$$\left(\frac{\alpha}{\alpha+\gamma} - \frac{\beta}{\beta+\delta}\right) \langle W| XC^{N-2} |V\rangle = -\left(N-3+\frac{1}{\beta+\delta}+\frac{1}{\alpha+\gamma}\right) \langle W| XXC^{N-3} |V\rangle . \tag{5.1.41}$$

Of course, checking when $(L_x - R_x) \langle W| XXC^{x-3}DC^{N-x-1} |V\rangle = 0$ results in an analogous expression as above, but with an extra X and $N-4$ explicitly instead of $N-3$. Recalling (5.1.31), one can clearly see that $X = x \in \mathbb{R}\backslash\{0\}$ is a very natural choice. Moreover, looking at the expression for the mean (5.1.34) we have that under this choice x would be a free variable. Furthermore, the choice $x = -1$ (and therefore $\mathcal{X} = (1,-1)^\dagger$) is the canonical choice for the algebra for the (linear) *SSEP*. To our knowledge, two-site correlations for this model were first computed through the *MPA* in [28] $-$ to where we refer the reader for the conclusion of the computations above (with slightly different parameters). Our goal with this section is to state a quite simple observation and make the reader familiar with the computations. Let us take $X_1 = x \in \mathbb{R}\backslash\{0\}$ and in-out rates as functions of the size of our system, as in (5.1.33). Then, note that the correlation for a system with $N-1$ sites is a function of systems with a smaller number of sites. Consider the quantity

$$< (\eta(1)(1-\eta(2)) - (1-\eta(1))\eta(2))\, \eta(N-1) >^w_{N-1} . \tag{5.1.42}$$

We will denote by $\underline{\langle W|}$ (resp. $\underline{|V\rangle}$) if we use our left (resp. right) relations to compute a local configuration at the left (resp. right) boundary. In this way, we must have

$$\underline{\langle W|}(DE-ED)C^{N-4}D\,|V\rangle = \langle W|\,(DE-ED)C^{N-4}D\underline{|V\rangle}. \tag{5.1.43}$$

Recalling our algebra (5.1.24), now we have $DE - ED = xC$. Therefore we compute:

$$\underline{\langle W|}(DE-ED)C^{N-4}D\,|V\rangle = x < \eta(N-2) >^w_{N-2} = x\left(\frac{\beta}{\beta+\delta}Z_{N-2} + \frac{x(N-1)^\theta}{\beta+\delta}Z_{N-3}\right), \tag{5.1.44}$$

while

$$\langle W| (DE-ED)C^{N-4}D\underline{|V\rangle} = \frac{\beta}{\beta+\delta} < \eta(1)(1-\eta(2)) - (1-\eta(1))\eta(2) >^w_{N-1} + \\ + \frac{xN^\theta}{\beta+\delta} < \eta(1)(1-\eta(2)) - (1-\eta(1))\eta(2) >^w_{N-2} = \frac{\beta}{\beta+\delta}xZ_{N-2} + \frac{x^2N^\theta}{\beta+\delta}Z_{N-3}. \tag{5.1.45}$$

We conclude that

$$\underline{\langle W|}(DE-ED)C^{N-4}D\,|V\rangle \neq \langle W|\,(DE-ED)C^{N-4}D\underline{|V\rangle} \tag{5.1.46}$$

for $\theta \neq 0$. Nevertheless, for large N we clearly have that $(N-1)^\theta \approx N^\theta$ thus the error is negligible, but still, it removes one of the main features of the *MPA* with slow/fast boundaries: *exact* solutions. The big issue lies in computing higher order correlations, thus making this error not negligible.

From [28] we know that the *MPA* solution for (5.1.38) *exactly* solves the discrete PDE for the correlation between two sites induced by the model, which has a *unique* solution − also in accordance with [9]. Both the *MPA* solution and the derivation of the PDE, however, were computed with N^θ fixed through all the computations, which lead to the correct result. Making N^θ completely dependent of the size of the system through the computations, however, and we'll have a small inconsistency. Our "solution" is to simply let X be a matrix. In this way one cannot decrease the size of the system and X behaves as a "ghost particle", thus fixing the boundary rates. Although this trick works for the well studied *linear SSEP*, for other non algebraically solvable models, or by solving the problem through the representation, one must have in mind that $N^{-\theta}$ is in fact *fixed*. The problem here lies in knowing how to formally fix this parameter, connecting with the mathematical framework, which we were not able to. Proceeding inductively, it easy to see from (5.1.31) that we have

$$Z_{N-1} = \langle W|\,X^{N-1}\,|V\rangle \left(\frac{\alpha}{\alpha+\gamma} - \frac{\beta}{\beta+\delta}\right)^{-N} \prod_{i=0}^{N-1} \left(i-1+\frac{N^\theta}{\beta+\delta}+\frac{N^\theta}{\alpha+\gamma}\right). \tag{5.1.47}$$

5.2 Extension of the methodology

We consider essentially the same base and framework as in (5.1.1), with a slightly different cancelation mechanism: at the boundaries we associate different matrices to the local configuration, and at the bulk we consider the usual cancelation mechanism with auxiliary vectors. The difference lies in not considering any auxiliary vector at the boundaries, thus letting the cancelation be more "natural". In this way, our *MPA formulation* for the invariant *measure* takes the form of the *ordered* product

$$\mu^w_{N-1}(\eta) = \langle W|\,(\eta_1 D_L + (1-\eta_1 E_L)) \times \prod_{i=2}^{N-2} [(1-\eta_i)E + \eta_i D] \times (\eta_{N-1}D_R + (1-\eta_{N-1}E_R))\,|V\rangle\,, \tag{5.2.1}$$

where the *boundary matrices* $D_{L,R}, E_{L,R}$ can be different from the *bulk matrices* D, E. Dividing by the normalization constant,

$$Z_{N-1} := \langle W|\,(D_L+E_L)C^{N-3}(D_R+E_R)\,|V\rangle\,, \tag{5.2.2}$$

we have the invariant *probability measure* μ_{ss}^N. The exact reason for this will be explained shortly. We let each boundary act on 2 sites, thus the jump operator M in (5.1.6) now takes the form

$$M = \sum_{k=1}^{N-2} m_{k,k+1} + B_{1,2}^L + B_{N-2,N-1}^R, \tag{5.2.3}$$

and the invariance condition (5.1.9) is again satisfied if $\partial_t |P(t)\rangle = M |P(t)\rangle = 0$. We defined above the boundary operators as $B_{1,2}^L := b_L \otimes 1^{\otimes N-3}$, and similar for the right, where b_L acts on $\mathbb{C} \otimes \mathbb{C}$. We force this condition by applying the telescopic rules in Definition 5.1.7 in the bulk only, and forcing the remaining terms to cancel with each boundary. Thus, letting (5.1.13) act on D, E matrices only, the remaining terms from the bulk are

$$-\langle W| X_L \otimes \mathcal{X} \otimes X^{\otimes N-4} \otimes X_R |V\rangle \quad \text{and} \quad \langle W| X_L \otimes X^{\otimes N-4} \otimes \mathcal{X} \otimes X_R |V\rangle . \tag{5.2.4}$$

Letting the term in left, on the previous display, cancel with $\langle W| (b_L + m) X_L \otimes X \otimes \cdots |V\rangle$ and the terms in the right, on the previous display, cancel with $\langle W| \cdots \otimes X \otimes X_R (b_R + m) |V\rangle$, we have the rules

$$\begin{aligned} &mX \otimes X = X \otimes \mathcal{X} - \mathcal{X} \otimes X, \\ &0 = \langle W| ((b_L + m) X_L \otimes X - \langle W| X_L \otimes \mathcal{X}), \\ &0 = (\mathcal{X} \otimes X_R + (b_R + m) X \otimes X_R) |V\rangle . \end{aligned} \tag{5.2.5}$$

5.2.1 Induced Algebra

As already seen in the previous section, the choice $X_1 = -X_2$ is enough to guarantee the algebraic solvability of the bulk induced by the *SSEP* dynamics. In this way, we will let $X_2 \equiv X = -X_1$ and fix our bulk algebra as in (5.1.24):

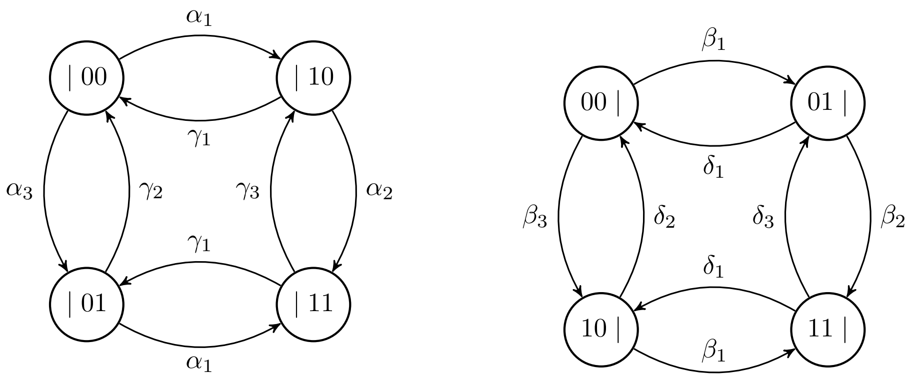
$$[E, X] = [D, X] = 0, \quad [D, E] = XC. \tag{5.2.6}$$

This way, we write the boundary algebra (5.2.5) more explicitly as:

$$0 = \langle W| \left[(m + b_L) \begin{pmatrix} E_L \\ D_L \end{pmatrix} \otimes \begin{pmatrix} E \\ D \end{pmatrix} - \begin{pmatrix} E_L \\ D_L \end{pmatrix} \otimes \begin{pmatrix} -X \\ X \end{pmatrix} \right], \tag{5.2.7}$$

$$0 = \left[(m + b_R) \begin{pmatrix} E \\ D \end{pmatrix} \otimes \begin{pmatrix} E_R \\ D_R \end{pmatrix} + \begin{pmatrix} -X \\ X \end{pmatrix} \otimes \begin{pmatrix} E_R \\ D_R \end{pmatrix} \right] |V\rangle . \tag{5.2.8}$$

We consider the most general boundary dynamics acting on two sites, in order to have a better control regarding under which choice of parameters our method works:



(5.2.9)

where we wrote $|\cdot$ and $\cdot|$ as the left and right reservoirs, respectively. The local jump operator acting in the bulk, m, is as in (5.1.5), but now the boundary jump operators are:

$$b_L = \begin{pmatrix} -(\alpha_1+\alpha_3) & \gamma_2 & \gamma_1 & 0 \\ \alpha_3 & -(\alpha_1+\gamma_2) & 0 & \gamma_1 \\ \alpha_1 & 0 & -(\alpha_2+\gamma_1) & \gamma_3 \\ 0 & \alpha_1 & \alpha_2 & -(\gamma_1+\gamma_3) \end{pmatrix} \tag{5.2.10}$$

$$b_R = \begin{pmatrix} -(\beta_1+\beta_3) & \delta_1 & \delta_2 & 0 \\ \beta_1 & -(\beta_2+\delta_1) & 0 & \delta_3 \\ \beta_3 & 0 & -(\beta_1+\delta_2) & \delta_1 \\ 0 & \beta_2 & \beta_1 & -(\delta_1+\delta_3) \end{pmatrix}. \tag{5.2.11}$$

Addicionally, we will focus on the *slow boundary* case, as in (5.1.33). Since all entries on b_L, b_R are aditive on the rates parameters, one can simply consider the matrices $b_L^{(\theta)} := N^{-\theta} b_L, b_R^{(\theta)} := N^{-\theta} b_R$.

5.2.2 Consistency

The objective of this section is to find under which conditions our algebra is consistent. As a byproduct, we will show that a closed expression for the normalization constant, $Z_{N-1}(\theta) := \langle W|\, C_L C^{N-3} C_R \,|V\rangle$, exists, where $C_L := D_L + E_L$ and $C_R := D_R + E_R$. Given the complexity of the rules (5.2.7) and (5.2.8), we found more convenient to work under a different "basis". From the previous section, we know that the boundaries are algebraically solvable if we can transform "particles" D into functions of the normalization constant $-$ more specifically, in C matrices. In this way, we will rewrite our rules as functions of C, as much as possible. Since our boundary dynamics acts in *two* sites each, it is natural to think that the two-sites correlation will play an important role. Moreover, note that $C_L D = DC_L - D_L E - E_L D$ (and similar for the right). In this way, we considered $C_L C, D_L C, D_L D$ and $D_L E - E_L D$ as our fundamental quantities (and similar for the right). Thus, consider the following substituitions on the boundary relations

$$[D,E] = CD - DC, \qquad EE = CC - DC - CD + DD, \tag{5.2.12}$$

$$ED = DC - [D,E] - DD, \qquad DE = DC - DD \tag{5.2.13}$$

and defining $\triangle_L = D_L E - E_L D$ and $\triangle_R = DE_R - ED_R$, we arrive at the relations

$$\langle W| \left[(m_L^* + b_L^{*,\theta}) \begin{pmatrix} C_L C \\ D_L C \\ D_L D \\ \triangle_L \end{pmatrix} - \begin{pmatrix} -(C_L - D_L)X \\ (C_L - D_L)X \\ -D_L X \\ D_L X \end{pmatrix} \right] = 0 \tag{5.2.14}$$

and similar for the right, with new matrices $b_L^{*,\theta}, b_R^{*,\theta}$, dependent of θ, as in the previous section (these matrices can be found in (D.2)). By either solving the boundary systems above as a linear system on the boundary terms of order 2, or by setting the common terms of each relation to be equal until we get a final relation and then replacing back, that is, for example (for the left):

- Let $D_L D$ be on the left hand-side of all expressions;
- Equate them two by two (first with second, second with third, third with fouth);
- Now we have 3 expressions instead of the initial four. Do the same for (say) $\triangle_L$;
- Now we have 2 expressions. Doing the same for (say) $D_L C$ shows that these two are already equal;
- Replace back $D_L C$ into one expression for $\triangle_L$ (note that they are all the same);
- Replace back $\triangle_L$ into $D_L D$ and now these three expressions are enough.

Our rules can be reduced to:

$$\langle W| D_L C = \langle W| d_1^L(N^0) C_L C + d_2^L(N^\theta) D_L X + d_3^L(N^\theta) C_L X, \tag{5.2.15}$$

$$\langle W| \triangle_L = \langle W| t_1^L(N^{-\theta}) C_L C + t_2^L(N^0) D_L X + t_3^L(N^0) C_L X, \tag{5.2.16}$$

$$\langle W| D_L D = \langle W| f_1^L(N^0) C_L C + f_2^L(N^\theta) D_L X + f_3^L(N^\theta) C_L X \tag{5.2.17}$$

and for the right

$$\begin{aligned} CD_R |V\rangle &= d_1^R(N^0) CC_R + d_2^R(N^\theta) XD_R + d_3^R(N^\theta) XC_R |V\rangle, \\ \triangle_R |V\rangle &= t_1^R(N^{-\theta}) CC_R + t_2^R(N^0) XD_R + t_3^R(N^0) XC_R |V\rangle, \\ DD_R |V\rangle &= f_1^R(N^0) CC_R + f_2^R(N^\theta) XD_R + f_3^R(N^\theta) XC_R |V\rangle, \end{aligned} \tag{5.2.18}$$

where the exponent on N denotes the order of the coefficient, for example, $d_1^{\cdot}(N^0) = \mathcal{O}(c)$, $d_2^{\cdot}(N^\theta) = \mathcal{O}(N^\theta)$, and so on. To reduce our rules, we programmed a *Mathematica* routine. The script can be found in ***here***. Moreover, the coefficients vanish under the following choice of parameters:

$$\begin{aligned} t_2^L &= 0 \Leftrightarrow \alpha_1+\gamma_1=0 \quad \vee \quad \alpha_2+\gamma_3=\gamma_2+\alpha_3 \\ t_3^L &= 0 \Leftrightarrow \alpha_1+\gamma_1=0 \quad \vee \quad \alpha_1+\alpha_2+\gamma_1+\gamma_3=0 \\ d_2^L &= 0 \Leftrightarrow \alpha_2+\gamma_3=\gamma_2+\alpha_3. \end{aligned} \tag{5.2.19}$$

For the right boundary the roots are analogous, considering the substitutions $(\alpha,\gamma)\mapsto(\beta,\delta)$. We will frequently write, for example, α, to denote $\{\alpha_1,\alpha_2,\alpha_3\}$, unless it is not clear from the context. The other coefficients vanish under expressions dependent on θ (for more details see (D.1)).

We will refer to the boundary algebra through the boundary coefficients in the following way:

$$\mathcal{A}_L^\theta\begin{pmatrix}\alpha\\ \gamma\end{pmatrix}=\begin{pmatrix}d_1^L & d_2^L & d_3^L\\ t_1^L & t_2^L & t_3^L\\ f_1^L & f_2^L & f_3^L\end{pmatrix} \quad \text{and} \quad \mathcal{A}_R^\theta\begin{pmatrix}\beta\\ \delta\end{pmatrix}=\begin{pmatrix}d_1^R & d_2^R & d_3^R\\ t_1^R & t_2^R & t_3^R\\ f_1^R & f_2^R & f_3^R\end{pmatrix}. \tag{5.2.20}$$

Thus, on this basis our boundary algebra is fully characterized (with some abuse of notation) by

$$\langle W|\begin{pmatrix}D_LC\\ \triangle_L\\ D_LD\end{pmatrix}=\langle W|\,\mathcal{A}_L^\theta\begin{pmatrix}C_LC\\ D_LX\\ C_LX\end{pmatrix}, \qquad \begin{pmatrix}CD_R\\ \triangle_R\\ DD_R\end{pmatrix}|V\rangle=\mathcal{A}_L^\theta\begin{pmatrix}CC_R\\ XD_R\\ XC_R\end{pmatrix}|V\rangle. \tag{5.2.21}$$

Example 5.2.1. Under this local "basis", the *linear SSEP* boundary algebra, studied on the previous section, is expressed as

$$\mathcal{A}_L^{\theta=0}\begin{pmatrix}\alpha & 0 & 0\\ \gamma & 0 & 0\end{pmatrix}=\begin{pmatrix}\frac{\alpha}{\alpha+\gamma} & 0 & -\frac{1}{\alpha+\gamma}\\ 0 & 0 & 1\\ \left(\frac{\alpha}{\alpha+\gamma}\right)^2 & -\frac{1}{\alpha+\gamma} & -\frac{\alpha(\alpha+\gamma+1)}{(\alpha+\gamma)^2}\end{pmatrix}, \quad \mathcal{A}_R^{\theta=0}\begin{pmatrix}\beta & 0 & 0\\ \delta & 0 & 0\end{pmatrix}=\begin{pmatrix}\frac{\beta}{\beta+\delta} & 0 & \frac{1}{\beta+\delta}\\ 0 & 0 & 1\\ \left(\frac{\beta}{\beta+\delta}\right)^2 & \frac{1}{\beta+\delta} & \frac{\beta(\beta+\delta+1)}{(\beta+\delta)^2}\end{pmatrix}. \tag{5.2.22}$$

The main interest of considering different boundary matrices is that $(t_1^{L,R},t_2^{L,R},t_3^{L,R})\neq(0,0,1)$, that is $\triangle_{L,R}\neq XC$. In this way, we allow the current at the windows (where the reservoirs act) to be different than the one flowing in the bulk. In order to make the reader believe that this formulation indeed describes the stationary state, we will start by checking the conditions on the discrete PDE for the mean, that should be satisfied by definition and, hopefully, without any aparent inconsistency. But first, let us define the *boundary recursions*:

$$\begin{aligned} &A_{N-3}=\langle W|\,D_LC^{N-3}C_R\,|V\rangle\,, \quad B_{N-3}=\langle W|\,C_LC^{N-3}D_R\,|V\rangle\,, \quad C_{N-3}=\langle W|\,C_LC^{N-3}C_R\,|V\rangle\,,\\ &D_{N-3}^L=\langle W|\,D_LDC^{N-4}C_R\,|V\rangle\,, \quad D_{N-3}^R=\langle W|\,C_LC^{N-4}DD_R\,|V\rangle\,,\\ &\triangle_{N-3}^L=\langle W|\,\triangle_LC^{N-4}C_R\,|V\rangle\,, \quad \triangle_{N-3}^R=\langle W|\,C_LC^{N-4}\triangle_R\,|V\rangle\,. \end{aligned} \tag{5.2.23}$$

Since the element X will appear on the following computations, we will write

$$X^n C_{N-3-n} := \langle W| \, C_L X^n C^{N-3-n} C_R \, |V\rangle \tag{5.2.24}$$

and simillar for the other recursions above. Note that X is completely free in the bulk (recall that $[D, X] = [E, X] = 0$), thus we can always rearrange it to the left/right. From direct application of the *Kolmogorov's equation* (2.0.32), one arrives at the following left boundary conditions

$$\begin{aligned}
&\alpha_1 N^{-\theta} < 1-\eta_1 >_{N-1} -\gamma_1 < \eta_1 >_{N-1} + < \eta_2 >_{N-1} - < \eta_1 >_{N-1} = 0 \\
&N^{-\theta}\alpha_2 < \eta_1(1-\eta_2) >_{N-1} -N^{-\theta}\gamma_2 < (1-\eta_1)\eta_2 >_{N-1} + N^{-\theta}\alpha_3 < (1-\eta_1)(1-\eta_2) >_{N-1} - \\
&- N^{-\theta}\gamma_3 < \eta_1\eta_2 >_{N-1} + < \eta_3 >_{N-1} -2 < \eta_2 >_{N-1} + < \eta_1 >_{N-1} = 0,
\end{aligned} \tag{5.2.25}$$

for the sites 1 and 2, respectively. Simplifying the conditions above we arrive at

$$\begin{aligned}
&- < \eta(1) >_{N-1} (N^{-\theta}(\alpha_1+\gamma_1)+1) + < \eta(2) >_{N-1} + N^{-\theta}\alpha_1 = 0 \\
&< \eta(1) >_{N-1} (N^{-\theta}(\alpha_2-\alpha_3)+1) - < \eta(2) >_{N-1} (N^{-\theta}(\gamma_2+\alpha_2)+2) + < \eta(3) >_{N-1} + \\
&+ < \eta(1)\eta(2) >_{N-1} N^{-\theta}(\gamma_2-\alpha_2+\alpha_3-\gamma_3) = 0.
\end{aligned} \tag{5.2.26}$$

Again, for the sites 1 and 2, respectively. Now note that $\langle W| \, C_L D = \langle W| \, D_L C - \triangle_L$ and $\langle W| \, C_L C D = \langle W| \, D_L C^2 - \triangle_L C - C_L C X$. In this way, taking the mean and writing in terms of the boundary recursions (5.2.23), then sending all particles to the left with the relations just mentioned, one arrives at

$$\begin{aligned}
&- A_{N-3} N^{-\theta}(\alpha_1+\gamma_1) - \triangle^L_{N-3} + N^{-\theta}\alpha_1 C_{N-3} = 0 \\
&A_{N-3} N^{-\theta}(\alpha_2 - 2\alpha_3 - \gamma_2) + \triangle^L_{N-3}(N^{-\theta}(\gamma_2+\alpha_3)+1) - XC_{N-4} + \\
&+ D^L_{N-3} N^{-\theta}(\gamma_2-\alpha_2+\alpha_3-\gamma_3) = 0.
\end{aligned} \tag{5.2.27}$$

Replacing our boundary algebra (5.2.15),(5.2.16) and (5.2.17), one has coefficients such that

$$l_1 C_{N-3} + l_2 X A_{N-4} + l_3 X C_{N-4} = 0 \quad \text{and} \quad s_1 C_{N-3} + s_2 X A_{N-4} + s_3 X C_{N-4} = 0 \tag{5.2.28}$$

for the sites 1 and 2, respectively, where

$$\begin{aligned}
&l_1 = N^\theta t^L_1 - \alpha_1 + (\alpha_1+\gamma_1) d^L_1, \quad l_2 = (\alpha_1+\gamma_1) d^L_2 + N^\theta t^L_2, \quad l_3 = N^\theta t^L_3 + (\alpha_1+\gamma_1) d^L_3, \\
&s_1 = d^L_1(\alpha_2-\gamma_2-2\alpha_3)N^{-\theta} + t^L_1(\gamma_2+\alpha_3)N^{-\theta} + f^L_1(\gamma_2-\alpha_2+\alpha_3-\gamma_3)N^{-\theta} + t^L_1 + \alpha_3 N^{-\theta}, \\
&s_2 = d^L_2(\alpha_2-\gamma_2-2\alpha_3)N^{-\theta} + t^L_2(\gamma_2+\alpha_3)N^{-\theta} + f^L_2(\gamma_2-\alpha_2+\alpha_3-\gamma_3)N^{-\theta} + t^L_2, \\
&s_3 = d^L_3(\alpha_2-\gamma_2-2\alpha_3)N^{-\theta} + t^L_3(\gamma_2+\alpha_3)N^{-\theta} + f^L_3(\gamma_2-\alpha_2+\alpha_3-\gamma_3)N^{-\theta} + t^L_3 - 1.
\end{aligned} \tag{5.2.29}$$

Although these coefficients are long, the interested reader can check that, indeed, $l_i = s_i = 0$ for any parameter (α, γ) and $\theta \geq 0$. For the right, the results are completely analogous. Now that we have checked that the formulation is correct, one needs to see whether the normalization constant is different than zero, that is, if there are no inconsistencies in our algebra. For that, let us first find an expression for the normalization.

Proposition 5.2.2. *For all $x \in \Lambda_N$:*

$$(R_x - L_x) < \eta_x >^w_{N-1} = 0 \Leftrightarrow A_{N-3} - B_{N-3} = \triangle^L_{N-3} + \triangle^R_{N-3} + (N-4)XC_{N-4}. \tag{5.2.30}$$

Proof. Computing from the left we have $L_x < \eta_x >^w_{N-1} = -(x-2)XC_{N-4} + A_{N-3} - \triangle^L_{N-3}$, and from the right: $R_x < \eta_x >^w_{N-1} = (N-(x+2))XC_{N-4} + B_{N-3} + \triangle^R_{N-3}$. Equating both expressions we get the result in the statement. Simillarly, letting $x = 1$ or $x = N-1$ in the left hand-side of (5.2.30), we get

$$\begin{aligned} &R_1 < \eta(1) >^w_{N-1} = \triangle^L_{N-3} + \triangle^R_{N-3} + (N-4)XC_{N-4} + B_{N-3} = A_{N-3} \\ &L_1 < \eta(N-1) >^w_{N-1} = -\triangle^L_{N-3} - \triangle^R_{N-3} - (N-4)XC_{N-4} + A_{N-3} = B_{N-3}. \end{aligned} \tag{5.2.31}$$

□

The main problem with finding a *closed expression* for C_{N-3}, that is, only in terms of C_{N-k} with $k > 0$, are the coefficients $d_2^{L,R}$ and $t_2^{L,R}$. The attentive reader might have noticed that these coefficients vanish *exactly* when there are no explicit correlations. Moreover, under these conditions, replacing our algebra it is immediate that (5.2.30) is a *geometric* recursion in terms of C − the same form as for the *linear SSEP*. However, this is the *only* case we found that brings inconsistencies to our algebra. Although this might seem counterintuitive − the simplest case being the one that does not work − with some thought this does make *some* sense. Recalling the normalization for the *linear SSEP* given in (5.1.47), we must have $\frac{\alpha}{\alpha+\gamma} \neq \frac{\beta}{\beta+\delta}$, otherwise $Z_{N-1} = 0$. For the more particular and more commonly studied choice of parameters, $\alpha + \gamma = \beta + \delta = 1$, we must have $\alpha \neq \beta$. Recalling that $d_2^L = t_2^L = 0 \Leftrightarrow \alpha_2 + \gamma_3 = \alpha_3 + \gamma_2$, we also have some sort of "equilibrium" in the non-linear rates, which makes the MPA to fail. To finally see when our algebra is consistent, note that the rules (5.2.15) and (5.2.17) are not always consistent. For that, we will deduce (5.2.17) from (5.2.15). Recalling the mentioned rules:

$$\langle W|\, D_L C = \langle W|\, d_1^L(N^0)C_L C + d_2^L(N^\theta)D_L X + d_3^L(N^\theta)C_L X, \tag{5.2.32}$$

$$\langle W|\, D_L D = \langle W|\, f_1^L(N^0)C_L C + f_2^L(N^\theta)D_L X + f_3^L(N^\theta)C_L X, \tag{5.2.33}$$

the idea is to reduce (5.2.15) and (5.2.17) by multiplying C through the right, then using commutators and the other relations to group all the "particles" in the left. We only do the computations for the left boundary, and supress the vector $\langle W|$ and L upperscript in the coefficients for simplicity, unless it is not clear from the context. In this way, note that

$$D_L C D = D_L D C - D_L C X = d_1 C_L C D + d_2 D_L D X + d_3 C_L D X. \tag{5.2.34}$$

Computing each term on the right hand-side of the last display

$$\begin{aligned} &C_L C D = (D_L C - \triangle_L - C_L X)C = ((d_1 - t_1)C_L C + (d_2 - t_2)D_L X + (d_3 - t_3 - 1)C_L X)C, \\ &D_L D X = f_1 C_L C X + f_2 D_L C X + f_3 C_L X X, \\ &C_L D X = D_L C X - \triangle_L X = D_L C X - t_1 C_L C X - t_2 D_L X X - t_3 C_L X X, \end{aligned} \tag{5.2.35}$$

we have

$$D_LDC - D_LCX = d_1\left((d_1-t_1)C_LCC + (d_2-t_2)D_LCX + (d_3-t_3-1)C_LCX\right) + \\ + d_2\left(f_1C_LCX + f_2D_LCX + f_3C_LXX\right) + d_3\left(D_LCX - t_1C_LCX - t_2D_LXX - t_3C_LXX\right). \tag{5.2.36}$$

Rearranging the terms:

$$D_LDC = C_LCC\left(d_1(d_1-t_1)\right) + C_LCX\left(d_1(d_3-t_3-1) + f_1d_2 - d_3t_1\right) + \\ +C_LXX\left(d_2f_3 - d_3t_3\right) + D_LCX\left(d_1(d_2-t_2) + d_2f_2 + d_3 + 1\right) - D_LXXd_3t_2. \tag{5.2.37}$$

But by (5.2.17) we must have the last term equal to $f_1C_LCC + f_2D_LCX + f_3C_LCX$. Equating the expression in the previous display to the one we just mentioned, and rearranging the terms we have

$$C_LCC\left(f_1 - d_1(d_1-t_1)\right) = C_LCX\left(d_1(d_3-t_3-1) + f_1d_2 - d_3t_1 - f_3\right) + \\ + C_LXX\left(d_2f_3 - d_3t_3\right) + D_LCX\left(d_1(d_2-t_2) + d_2f_2 + d_3 + 1 - f_2\right) - D_LXXd_3t_2. \tag{5.2.38}$$

Now it is clear the inconsistency by setting $d_2^L = t_2^L = 0$. If these coefficients vanish, then the expression on the previous display is a closed recursion for the normalization, but inconsistent with (5.2.30). While we could have that, after replacing our algebra in D_LCX, all the coefficients vanish, this is not the case and we will always have to make further restrictions in our parameter space. Let $d_2^L, t_2^L \neq 0$, and most of our problems are solved. By replacing our algebra in D_LCX, one has an expression for D_LXX as a function of C_LCC, C_LCX and C_LXX. This is good because now we can successively replace this back in (5.2.15) and (5.2.16) to get an expression for D_LC and $\triangle_L$ as a function of the normalization only (modulo X matrices). Replacing this in our expression for the normalization (5.2.30), and doing completely analogous computations for the right boundary, one can see that $C_{N-3} \equiv Z_{N-1}$ is a *second order recurrence*. While this means that we cannot fully solve the MPA problem algebraically, we can get good enough approximations for any physical quantity by iterating the recurrence. Furthermore, we can solve the continuous PDE *exactly* by simply taking the limit, and noticing that we have $\frac{C_{N-4}}{C_{N-3}} = \mathcal{O}(N^{-1})$ for $\theta \in [0,1]$, and $\frac{C_{N-4}}{C_{N-3}} = \mathcal{O}(N^{-a})$ with $a > 1$ for $\theta > 1$, just as for the *linear SSEP*.

Chapter 6

Conclusions and future work

As seen in Chapter 3 and 4, the *slow* reservoirs impose not much difficulty when $\theta \geq 1$. When $\theta < 1$, however, these arguments do not work. In Chapter 3, we saw in the heuristics for the hydrodynamic limit, Section 3.2, that $\theta < 1$ is a problem, since the martingale would "explode" as $N \to \infty$. We can work around this by choosing a more appropriate test function and notion of weak solution. In the formal proof, we see the same problem when showing the tightness of the sequence $\{\mathbb{Q}^N\}_{N\geq 1}$, by the end of Subsection 3.3.1, and when showing the Replacement Lemmas. In [3] it is shown that the sequence is *tight* for all $\theta \in [0, 1)$ and $K = 1$, where the trick lies in taking a different test function H that, instead of only being in C^2, it has now compact support. The same argument works for general K, but then we run into the problem again in the proof of the Replacement Lemmas$-$ to be more specific, in (A.0.27), for example, where we bound the *Dirichlet form*. A solution might be to exchange the measure to a more appropriate one, through the entropy inequality. Following on this direction, one could use the *MPA* formulated in Chapter 5, for $K = 2$. Nevertheless, we were successful in proving the Hydrodynamic Limit for $\theta \geq 1$, and in the treatment of the correlation terms.

For the *propagation of chaos*, from the moment that we compute the bounds for w'_π and $w''_{\pi,H}$ in terms of "full powers", in Proposition 4.4.5 and Lemma 4.4.2, it is clear that for $\theta < 1$ the exponents are too large to treat. For $\theta > 1$, as already mentioned, we are confident in the bound $c_n(\epsilon^{-2}t)^{-n\theta c^*}$ for the $v-$functions, since we were able to show it for $m = M$, that is, when we truncate the series of iterations for the $v-$functions.

To show the Hydrodynamic Limit for $\theta < 1$, we have a preliminary work where we consider the generator of a random walker that is *absorbed* in the boundaries, instead of reflected as in Chapter 4. This leads to the estimation of other quantities, after an application of Feynman-Kac's formula. We believe that the study of the article [24] will prove to be fruitfull in terms of ideas for the proof.

Regarding the Matrix Product Ansatz, we have not mentioned the existence of such matrices. In fact, a lot of work has been done in this direction, but it is still not ready *yet*. Our strategy was to adapt the *bulk* matrices, D and E, presented in [8], then apply the boundary relations to find the vectors $\langle W_0| := \langle W| E_L$, $\langle W_1| := \langle W| D_L$, and similar for the right. From [8], we were able to find 2 more matrices that satisfy the bulk condition $[D, E] = C$ (note that we considered $\theta = 0$ and $x = 1$). For one of them, we were able to find a reccursion relation for each boundary vector, thus showing existence. The problem lies in noticing (which we only did in a final stage of this thesis) that we may sum any constant b to each matrix, and the bulk relations still hold (for example, letting

$D' := 1b + D$), and this constant is, clearly, incorporated into our boundary vectors. Thus, one needs to show that either the constant is *free* (in the sense that it vanishes when computing any quantity), or fix it. Even later in the development of this thesis, we noticed in [5] that one may "force" this constant in order to have the correct result for the MPA, but with a catch: one needs the explicit solution of the homogenous discrete PDE that the empirical mean satisfies. The strategy is quite simple: explicitly compute the mean through the left by using the *representation*, then through the right, equate them both, and check what the constant b may be in order to match with the known stationary solution of the PDE. For $K = 2$ one can, however, explicitly solve the *continuous* homogenous equation (the problem lies in solving polynomials of degree 4, but one may choose the parameters appropriately to simplify the problem). Thus, our strategy is to compute the mean with the representation, take the limit $N \to \infty$ then check wether b vanishes or not. But again, we have a new problem. Due to our number of boundary rules and their complexity, and the infinite representation of the bulk matrices, the vectors have *infinite* entries different than zero, and we could not compute any quantity. Professor Gunter M. Schutz suggested two new matrices, that still need to be tested. Thus, there is plenty of work in this direction.

Regarding the MPA still, we remark that we considered the boundary to act in the *same* number of sites as the bulk, and we know that if the boundaries act in one site less than the bulk, the current methodology, in principle, should work, unless there are inconsistencies in the algebra for any non trivial choice of parameters. One could consider the opposite problem, and let the boundaries act in more sites than the bulk. To our knowledge, there has not been any work in this direction. While one could proceed exactly as we did through this thesis, it is clear that the rules will be completely chaotic. It would be interesting to consider our method as a particular case of a cancelation mechanism acting in the window that each reservoir acts on $-$similar to what it is done in the bulk. In this way, one may focus more in the rules induced by the auxiliary matrices, rather than all the terms in the boundary relations as we did. Moreover, we know that the *TASEP* (Totally Asymmetric Exclusion Process) is algebraically solvable (in fact, the TASEP was the first model solved through the MPA, in [8]), and also the *zero-range process*, as already mentioned. It would be interesting to see how consistent the boundary rules are with respect to the extension in this thesis, under this change of bulk dynamics.

Regardless of all the technical difficulties for $\theta < 1$, with respect to the Hydrodynamic Limit and the propagation of chaos; and the complexity of the MPA's representation, we were sucessfull in showing the Hydrodynamic Limit for a new and technically interesting model, and in studying regimes not yet studied in [24] as a particular case. Moreover, we deconstructed a difficult article and showed that the propagation of chaos still holds for our model; made corrections in a classic algebra, introduced new tools for an algebraic study, and extended a method in a direction not yet successfully extended since its formulation, which not only opens new doors for different extensions, but allows us to study large deviations and the Hydrostatic Limit with a simpler approach for models not studied yet. And for that, we are proud.

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Appendix A

Replacement Lemmas

In this section we prove the Replacements Lemmas that are needed in order to get the weak formulation of solutions to the hydrodynamic equation. As already mentioned, we will only make the proofs for $K = 2$, since the arguments for general K are analogous. We start with some definitions. We recall from Example 3.3.3 the definition of the *Bernoulli product measure*, $\nu^N_{\gamma(\cdot)}$, defined by the marginals

$$\nu^N_{\gamma(\cdot)}(\eta : \eta(x) = 1) = \gamma(\tfrac{x}{N}), \tag{A.0.1}$$

where $\gamma : [0,1] \to [0,1]$ is a measurable profile. Note that the total probability of a configuration takes the form $\nu^N_{\gamma(\cdot)}(\eta) = \prod_{x\in\Lambda_N}(\gamma(\frac{x}{N}))^{\eta(x)}(1-\gamma(\frac{x}{N}))^{1-\eta(x)}$, and if γ is a constant function then ν^N_γ is invariant for the change of variables,that is, $\nu^N_\gamma(\eta^{x,x+1}) = \nu^N_\gamma(\eta)$.

Definition A.0.1 (Dirichlet form)**.** For a probability measure μ on Ω_N and a density $f : \Omega_N \to \mathbb{R}$ with respect to μ, the Dirichlet form $\mathcal{L}_N$ of f is defined as $\langle\sqrt{f}, -\mathcal{L}_N\sqrt{f}\rangle_\mu$. In our case we have

$$\langle\sqrt{f}, -\mathcal{L}_N\sqrt{f}\rangle_\mu = \langle\sqrt{f}, -\mathcal{L}_{N,0}\sqrt{f}\rangle_\mu + \tfrac{\kappa}{N^\theta}\langle\sqrt{f}, -L^L_{N,b}\sqrt{f}\rangle_\mu + \tfrac{\kappa}{N^\theta}\langle\sqrt{f}, -\mathcal{L}^{NL}_{N,b}\sqrt{f}\rangle_\mu, \tag{A.0.2}$$

where we recall that $\langle f, g\rangle_\mu = \int_{\Omega_N} f(\eta)g(\eta)\,d\mu$, for all functions $f, g : \Omega_N \to \mathbb{R}$.

Our first computation is a comparison between the Dirichlet form just defined and the next quantity, also known in the literature as *careé du champ*:

$$D_N(\sqrt{f},\mu) := D_{N,0}(\sqrt{f},\mu) + \frac{\kappa}{N^\theta}D^L_{N,b}(\sqrt{f},\mu) + \frac{\kappa}{N^\theta}D^{NL}_{N,b}(\sqrt{f},\mu), \tag{A.0.3}$$

where

$$D_{N,0}(\sqrt{f},\mu) \;:=\; \sum_{x=1}^{n-2}\int_{\Omega_N}\left[\sqrt{f(\eta^{x,x+1})} - \sqrt{f(\eta)}\right]^2 d\mu,$$

$$D^L_{N,b}(\sqrt{f},\mu) = D^L_{N,-}(\sqrt{f},\mu) + D^L_{N,+}(\sqrt{f},\mu) \quad \text{and} \quad D^{NL}_{N,b}(\sqrt{f},\mu) = D^{NL}_{N,-}(\sqrt{f},\mu) + D^{NL}_{N,+}(\sqrt{f},\mu), \tag{A.0.4}$$

with

$$
\begin{aligned}
D^{L}_{N,-}(\sqrt{f},\mu) =& \frac{\kappa}{N^\theta}\int (\alpha_1(1-\eta(1))+\gamma_1\eta(1))[\sqrt{f(\eta^1)}-\sqrt{f(\eta)}]^2\,d\mu,\\
D^{L}_{N,+}(\sqrt{f},\mu) =& \frac{\kappa}{N^\theta}\int (\beta_1(1-\eta(N-1))+\delta_1\eta(N-1))[\sqrt{f(\eta^{N-1})}-\sqrt{f(\eta)}]^2\,d\mu,
\end{aligned}
\tag{A.0.5}
$$

and

$$
\begin{aligned}
D^{NL}_{N,-}(\sqrt{f},\mu) =& \frac{\kappa}{N^\theta}\int (\alpha_2\eta(1)(1-\eta(2))+\gamma_2(1-\eta(1))\eta(2))[\sqrt{f(\eta^2)}-\sqrt{f(\eta)}]^2\,d\mu,\\
D^{NL}_{N,+}(\sqrt{f},\mu) =& \frac{\kappa}{N^\theta}\int (\beta_2\eta(N-1)(1-\eta(N-2))+\delta_2(1-\eta(N-1))\eta(N-2))[\sqrt{f(\eta^{N-2})}-\sqrt{f(\eta)}]^2\,d\mu.
\end{aligned}
\tag{A.0.6}
$$

The reason for us to introduce these quantities is simple. Looking at the statements of the Replacement Lemmas A.0.6 and A.0.5, we will use the entropy inequality (C.3.2) to "exchange" the measure μ^N to a simpler one, while controling the entropy between both. In this way, we want to use a measure where the entropy of the measure induced by our process with respect to the new one is going to be small enough, and product Bernoulli with constant profile, as we will see, is enough for $\theta \geq 1$. Once applied the entropy inequality, we can use Feynman-Kac's inequality, thus needing to estimate the quantities defined above.

We claim that for $\theta \geq 1$ and for $\gamma : [0,1] \to [0,1]$ a constant profile equal to, for example, γ, the following bound holds

$$
\langle \mathcal{L}_N\sqrt{f},\sqrt{f}\rangle_{\nu^N_\gamma} \lesssim -D_N(\sqrt{f},\nu^N_\gamma)+\mathcal{O}(N^{-\theta}). \tag{A.0.7}
$$

The following estimates are not difficult to obtain and rely essentialy on writing the terms $\langle \mathcal{L}_{N,\cdot}\sqrt{f},\sqrt{f}\rangle_{\nu^N_\gamma}$ as its half plus its half, summing and subtracting the appropriate terms in one of the halfs to get $-D_N(\sqrt{f},\nu^N_\gamma)$ plus a new term. Then, group the remaining half and the new term, and make an appropriate change of variables to take advantage of the particularity of ν^N_γ being invariant for the change of variables $\eta \mapsto \eta^{x,x+1}$ (for the bulk), or replace $\nu^N_\gamma(\eta) \mapsto \nu^N_\gamma(\eta^x)$ with a small error, for the boundary terms. Thus, we will refer the reader to C.2.2 for the full statement, and only refer at the moment that we know that:

$$
\langle \mathcal{L}_{N,0}\sqrt{f},\sqrt{f}\rangle_{\nu^N_\gamma} = -D_{N,0}(\sqrt{f},\nu^N_\gamma) \quad\text{and}\quad \langle \mathcal{L}^L_{N,-}\sqrt{f},\sqrt{f}\rangle_{\nu^N_\gamma} \lesssim -D^L_{N,-}(\sqrt{f},\nu^N_\gamma)+\mathcal{O}(N^{-\theta}). \tag{A.0.8}
$$

Thus, it is only necessary to control the contribution from $\mathcal{L}^{NL}_{N,b}$. For that purpose we recall the following lemma from [4]:

Lemma A.0.2. *Let $T : \eta \in \Omega_N \to T(\eta) \in \Omega_N$ be a transformation, and $c : \eta \to c(\eta)$ be a positive local function. Let f be a density with respect to a probability measure μ on Ω_N. Then, we have that*

$$
\begin{aligned}
&\int c(\eta)[\sqrt{f(T(\eta))}-\sqrt{f(\eta)}]\;\;\sqrt{f(\eta)}\,d\mu \lesssim -\int c(\eta)\left[\sqrt{f(T(\eta))}-\sqrt{f(\eta)}\right]^2\,d\mu+\\
&+\int \frac{1}{c(\eta)}\left[c(\eta)-c(T(\eta))\frac{\mu(T(\eta))}{\mu(\eta)}\right]^2\left[\sqrt{f(T(\eta))}+\sqrt{f(\eta)}\right]^2 d\mu.
\end{aligned}
\tag{A.0.9}
$$

From this lemma we have

$$\int \alpha_2\eta(1)(1-\eta(2)+\gamma_2(1-\eta(1))\eta(2))[\sqrt{f(\eta^2)}-\sqrt{f(\eta)}]\sqrt{f(\eta)}d\nu_\gamma^N \tag{A.0.10}$$

$$\lesssim -D_{N,-}^{NL}(\sqrt{f},\nu_\gamma^N)+\int N^{-\theta}\left[1-\frac{\nu_\gamma^N(\eta^2)}{\nu_\gamma^N(\eta)}\right]^2[\sqrt{f(\eta^2)}-\sqrt{f(\eta)}]^2d\nu_\gamma^N, \tag{A.0.11}$$

where we bounded $\alpha_2\eta(1)(1-\eta(2))+\gamma_2(1-\eta(1))\eta(2)$ and $\alpha_2\eta(1)\eta(2)+\gamma_2(1-\eta(1))(1-\eta(2))$ by a constant. Since ν_γ^N is Bernoulli with constant marginals, we have (A.0.7).

Remark A.0.3. For the general case $K>2$ the only difference lies in the fact that we will have to consider $\mathcal{L}_{N,-}^{NL}\equiv\sum_{x\geq 2}\mathcal{L}_{N,-}^{NL,x}$ where for each x we have the non linear dynamics acting at the sites $2,\ldots,x$. Just as in the case studied above we have $\mathcal{L}_{N,-}^{NL}\equiv\mathcal{L}_{N,-}^{NL,2}$, with the aforementioned notation. In this way, the error from the non linear dynamics would be of order $(K-1)N^{-\theta}$.

The next lemma will allow us to prove one of the Replacement Lemmas that is needed in the proof of hydrodynamics.

Lemma A.0.4. *Let $x<y\in\Lambda_N$ and let $\varphi:\Omega\to\Omega$ be a positive and bounded function which satisfies $\varphi(\eta)=\varphi(\eta^{z,z+1})$ for any $z=x,\cdots,y-1$. For any density f with respect to ν_γ^N and any positive constant A, it holds that*

$$\left|\langle\varphi(\eta)(\eta(x)-\eta(y)),f\rangle_{\nu_\gamma^N}\right|\lesssim \tfrac{1}{A}D_N(\sqrt{f},\nu_\gamma^N)+A.$$

Proof. Note that $\eta(x)-\eta(y)=\sum_{z=x}^{y-1}(\eta(z)-\eta(z+1))$. Summing and subtracting $\sum_{z=x}^{y-1}f(\eta^{z,z+1})$ we have:

$$\begin{aligned}|\langle\varphi(\eta)(\eta(x)-\eta(y)),f\rangle_{\nu_\gamma^N}|\leq\frac{1}{2}\sum_{z=x}^{y-1}\left|\int\varphi(\eta)(\eta(z)-\eta(z+1))[f(\eta)-f(\eta^{z,z+1})]\,d\nu_\gamma^N\right|\\ +\frac{1}{2}\sum_{z=x}^{y-1}\left|\int\varphi(\eta)(\eta(z)-\eta(z+1))[f(\eta)+f(\eta^{z,z+1})]\,d\nu_\gamma^N\right|.\end{aligned} \tag{A.0.12}$$

Note that since φ satisfies $\varphi(\eta)=\varphi(\eta^{z,z+1})$ for any $z=x,\cdots,y-1$, by a change of variables, we conclude that the last term in the previous display is equal to zero:

$$\int\varphi(\eta)(\eta(z)-\eta(z+1))f(\eta)d\nu_\gamma^N=\sum_{\eta\in\Omega_N}\varphi(\eta)(\eta(z)-\eta(z+1))f(\eta)\nu_\gamma^N(\eta) \tag{A.0.13}$$

$$=\sum_{\xi\in\Omega_N}\varphi(\xi)(\xi(z+1)-\xi(z))f(\xi^{z,z+1})\frac{\nu_\gamma^N(\xi^{z,z+1})}{\nu_\gamma^N(\xi)}\nu_\gamma^N(\xi), \tag{A.0.14}$$

since ν_γ^N has constant marginals we have that $\nu_\gamma^N(\xi^{z,z+1})/\nu_\gamma^N(\xi)=1$ and the last expression on the previous display simplifies to

$$-\sum_{\xi\in\Omega_N}\varphi(\xi)(\xi(z)-\xi(z+1))f(\xi^{z,z+1})\nu_\gamma^N(\xi). \tag{A.0.15}$$

Now, we treat the remaining term. Using the equality $(a-b)=(\sqrt{a}-\sqrt{b})(\sqrt{a}+\sqrt{b})$ and then Young's inequality, the first term at the right side of (A.0.12) is bounded from above by

$$A\sum_{z=x}^{y-1}\left|\int(\varphi(\eta)(\eta(z)-\eta(z+1)))^2\left(\sqrt{f(\eta^{z,z+1})}+\sqrt{f(\eta)}\right)^2d\nu_\gamma^N\right|+\frac{1}{A}\sum_{z=x}^{y-1}\left|\int\left(\sqrt{f(\eta)}-\sqrt{f(\eta^{z,z+1})}\right)^2d\nu_\gamma^N\right|. \tag{A.0.16}$$

Recalling the definition of $D_{N,0}(\sqrt{f},\nu_\gamma^N)$, clearly we have $D_{N,0}(\sqrt{f},\nu_\gamma^N)\leq D_N(\sqrt{f},\nu_\gamma^N)$. From the fact that φ is bounded, $|\eta(x)|\leq 1$ and f is a density, the term on the left-hand side of last expression is bounded from above by a constant times A. This ends the proof. □

Lemma A.0.5 (Replacement Lemma 1). *Let $\varphi:\Omega\to\Omega$ be a positive and bounded function which satisfies $\varphi(\eta)=\varphi(\eta^{z,z+1})$ for any $z=x,\cdots,y-1$. For any $t\in[0,T]$ we have that*

$$\limsup_{N\to+\infty}\mathbb{E}_{\mu_N}\left[\left|\int_0^t\varphi(\eta_{sN^2})(\eta_{sN^2}(x)-\eta_{sN^2}(y))ds\right|\right]=0. \tag{A.0.17}$$

Proof. The expectation above can be written as

$$\int\mathbb{E}_\eta\left[\left|\int_0^t\varphi(\eta_{sn^2})(\eta_{sN^2}(x)-\eta_{sN^2}(y))ds\right|\right]d\mu_N. \tag{A.0.18}$$

From entropy inequality (C.3.2), for any $B>0$, last display is bounded from above by

$$\frac{H(\mu_N|\nu_\gamma^N)}{BN}+\frac{1}{BN}\log\int e^{B\mathbb{E}_\eta\left[\left|\int_0^t\varphi(\eta_{sN^2})(\eta_{sN^2}(x)-\eta_{sN^2}(y))ds\right|\right]}d\nu_\gamma^N. \tag{A.0.19}$$

Now we use Jensen's inequality and bound the previous expression by

$$\frac{H(\mu_N|\nu_\gamma^N)}{BN}+\frac{1}{BN}\log\int\mathbb{E}_\eta\left[e^{BN\left|\int_0^t\varphi(\eta_{sN^2})(\eta_{sN^2}(x)-\eta_{sN^2}(y))ds\right|}\right]d\nu_\gamma^N, \tag{A.0.20}$$

which is equal to

$$\frac{H(\mu_N|\nu_\gamma^N)}{BN}+\frac{1}{BN}\log\mathbb{E}_{\nu_\gamma^N}\left[e^{BN\left|\int_0^t\varphi(\eta_{sN^2})(\eta_{sN^2}(x)-\eta_{sN^2}(y))ds\right|}\right]. \tag{A.0.21}$$

Now we apply the inequality $e^{|x|}\leq e^x+e^{-x}$ to be free of the absolute value inside the exponential. Noticing that

$$\limsup_{N\to+\infty}N^{-1}\log(a_N+b_N)=\max\{\limsup_{N\to+\infty}N^{-1}\log(a_N),\limsup_{N\to+\infty}N^{-1}\log(b_N)\},$$

we only need to work with the exponential with positive expoent. It is easy to see that $H(\mu_N|\nu_\gamma^N)\leq NC_\gamma$, since by (C.3.1)

$$H(\mu_N|\nu_\gamma^N)=\sum_{\eta\in\Omega_N}\mu(\eta)\log\frac{\mu(\eta)}{\nu_\gamma^N(\eta)}\leq\sum_{\eta\in\Omega_N}\mu(\eta)\log\frac{1}{\nu_\gamma^N(\eta)}. \tag{A.0.22}$$

Recalling that $\nu_\gamma^N(\eta)=\prod_{x\in\Lambda_N}\gamma^{\eta(x)}(1-\gamma)^{1-\eta(x)}$, let $\gamma_\wedge:=\gamma\wedge(1-\gamma)$. Then $\frac{1}{\nu_\gamma^N(\eta)}\geq\gamma_\wedge^{-N}$. Since the logarithm is an increasing function, we have

$$\sum_{\eta\in\Omega_N}\mu(\eta)\log\frac{1}{\nu_\gamma^N(\eta)}\leq N\sum_{\eta\in\Omega_N}\mu(\eta)\log\frac{1}{\gamma_\wedge}\leq NC_\gamma, \tag{A.0.23}$$

where we only needed to impose that $\gamma\neq 0,1$. By Feynman-Kac's inequality (C.4.1), expression (A.0.21) is bounded from above by

$$\tfrac{C_\gamma}{B}+t\sup_f\Big\{\langle\varphi(\eta)(\eta(x)-\eta(y)),f\rangle_{\nu_\gamma^N}+\tfrac{N}{B}\langle\mathcal{L}_N\sqrt{f},\sqrt{f}\rangle_{\nu_\gamma^N}\Big\}. \tag{A.0.24}$$

The supremum above is over densities f with respect to ν_α. By Lemma A.0.4, with the choice $A=\frac{B}{N}$ we have that

$$\Big|\langle\varphi(\eta)(\eta(x)-\eta(y)),f\rangle_{\nu_\gamma^N}\Big|\lesssim\tfrac{N}{B}D_N(\sqrt{f},\nu_\gamma^N)+\tfrac{B}{N}. \tag{A.0.25}$$

From (A.0.7) and the inequality above, the term on the right-hand side of (A.0.24), is bounded from above by $\frac{B}{N}+\frac{1}{N}$. Taking $N\to\infty$ and then $B\to+\infty$ we finish the proof. □

Lemma A.0.6 (Replacement Lemma 2). *Let $\psi:\Omega\to\Omega$ be a positive and bounded function which satisfies $\psi(\eta)=\psi(\eta^{z,z+1})$ for any $z=x+1,\cdots,x+\varepsilon N-1$. For any $t\in[0,T]$ and $x\in\Lambda_N$ such that $x\in\{1,\cdots,N-\varepsilon N-2\}$ we have that*

$$\limsup_{N\to+\infty}\mathbb{E}_{\mu_N}\left[\left|\int_0^t\psi(\eta_{sN^2})(\eta_{sN^2}(x)-\overrightarrow{\eta}^{\varepsilon N}_{sN^2}(x))ds\right|\right]=0. \tag{A.0.26}$$

Note that for $x\in\Lambda_N$ such that $x\in\{N-\varepsilon N-1,N-1\}$ the previous result is also true, but we replace in the previous expectation $\overrightarrow{\eta}^{\varepsilon N}_{sN^2}(x)$ by $\overleftarrow{\eta}^{\varepsilon N}_{sN^2}(x)$.

Proof. We present the proof for the case when $x\in\{1,\cdots,N-\varepsilon N-2\}$ since the other case is analogous. By applying the same arguments as in the proof of the previous theorem, we can bound the previous expectation by

$$\tfrac{C_\alpha}{B}+t\sup_f\Big\{\langle\psi(\eta)(\eta(x)-\overrightarrow{\eta}^{\varepsilon N}(x)),f\rangle_{\nu_\gamma^N}+\tfrac{N}{B}\langle\mathcal{L}_N\sqrt{f},\sqrt{f}\rangle_{\nu_\gamma^N}\Big\}. \tag{A.0.27}$$

where B is a positive constant. The supremum above is over densities f with respect to ν_γ^N. From the definition of $\overrightarrow{\eta}^{\varepsilon N}(x)$ in (3.3.38), first term in the supremum above can be written as

$$\frac{1}{\varepsilon N}\sum_{y=x+1}^{x+\varepsilon N}\langle\psi(\eta)(\eta(x)-\eta(y)),f\rangle_{\nu_\gamma^N}$$

By Lemma A.0.4 with the choice $A=\frac{B}{N}$ and from (A.0.7), the term on the right-hand side of (A.0.27), is again bounded from above by $\frac{B}{N}+\frac{1}{N}$. Taking $N\to\infty$ and then $B\to+\infty$ we are done. □

Appendix B

Energy Estimate

In this section we will show that the density ρ lives in a *Sobolev space*, almost surely in $t \in [0,T]$. The procedure is standard, and we will follow mostly [3] and [10]. We start by presenting some notation for this section.

- $\langle \cdot, \cdot \rangle_\mu$: inner product in $L^2([0,1])$ with respect to measure μ defined in $[0,1]$ and $\|\cdot\|_{L^2}$ the corresponding norm;
- $C^{m,n}([0,T] \times [0,1])$: set of functions m times differentiable on first variable and n times differentiable on the second variable with continuous derivatives;
- $C_c^{m,n}([0,T] \times [0,1])$: subset of $H \in C^{m,n}([0,T] \times [0,1])$ such that H_s has compact support in $(0,1)$;
- $C_c^m(0,1)$: set of m continuously differentiable real-valued functions defined on $(0,1)$ with compact support.

Definition B.0.1. The semi inner-product $\langle \cdot, \cdot \rangle_1$ on $C^\infty([0,1])$ is defined as:

$$\langle G, H \rangle_1 = \int_0^1 (\partial_v G)(v)(\partial_v H)(v) dv, \tag{B.0.1}$$

for $G, H \in C^\infty([0,1])$, and the corresponding semi-norm is denoted by $\|\cdot\|_1$.

Definition B.0.2. The *Sobolev space* on $(0,1)$, $\mathcal{H}^1(0,1)$ is the Hilbert space defined as the completion of $C^\infty(0,1)$ for the norm $\|\cdot\|_{\mathcal{H}^1}^2 := \|\cdot\|_{L^2}^2 + \|\cdot\|_1^2$, and its elements coincide *a.e.* with continuous functions. Moreover, we define $L^2(0,T;\mathcal{H}^1(0,1)) = \{f : [0,T] \longrightarrow \mathcal{H}^1 \mid \int_0^T \|f_s\|_{\mathcal{H}^1(0,1)}^2 ds < \infty\}$.

Now we show that the density ρ lives in the space $L^2(0,T;\mathcal{H}^1(0,1))$. This result can be summarized in the following proposition.

Proposition B.0.3. *Let $\mathbb{Q}$ be concentrated on paths $\pi_t(dv) = \rho_t(v)dv$. There, exists a function in $L^2([0,T] \times (0,1))$ denoted by $\partial_v \rho$ such that*

$$\int_0^T \int_0^1 (\partial_v G)(s,v)\rho(s,v) dv ds = -\int_0^T \int_0^1 G(s,v)(\partial_v \rho)(s,v) dv ds \tag{B.0.2}$$

for all $G \in C^{0,1}([0,T] \times (0,1))$.

To see this, for simplicity we consider the linear functional ℓ_ρ defined in $C_c^{0,1}([0,T]\times(0,1))$ by

$$\ell_\rho(G)=\int_0^T\int_0^1\partial_v G_s(v)\rho(s,v)\,dvds=\int_0^T\int_0^1\partial_v G_s(v)\,d\pi(s,v)ds. \tag{B.0.3}$$

By Proposition B.0.4, the proof of Proposition B.0.3 is simple.

Proof of Proposition B.0.3. Since the set of functions $H\in C^{0,1}([0,T]\times(0,1))$ is dense in $L^2([0,T]\times(0,1))$, and by Propposition B.0.4 ℓ is $\mathbb{Q}-a.s.$ bounded functional in $C^{0,1}([0,T]\times(0,1))$, one can extend this functional to $L^2([0,T]\times(0,1))$. As a consequence of the Riesz's Representation Theorem, there exists a function $H\in L^2([0,T]\times(0,1))$ such that for all $G\in C_c^{0,1}([0,T]\times(0,1))$

$$\ell_\rho(G)=-\int_0^T\int_0^1 G_s(v)H_s(v)dvds. \tag{B.0.4}$$

From this we conclude that $\rho\in L^2(0,T;\mathcal{H}^1(0,1))$. □

Proposition B.0.4. *There exist positive constants C and c such that*

$$\mathbb{E}\Big[\sup_{G\in C_c^{0,1}([0,T]\times(0,1))}\Big\{\ell_\rho(G)-c\|G\|_2^2\Big\}\Big]\le C<\infty, \tag{B.0.5}$$

where $\|G\|_2$ is the L^2 norm of a function $G\in L^2([0,T]\times(0,1))$.

Proof. By density and by the *Monotone Convergence Theorem* we only need to show that for a countable dense subset $\{G_m\}_{m\in\mathbb{N}}$ on $C_c^{0,2}([0,T]\times(0,1))$ holds

$$\mathbb{E}\left[\max_{k\le m}\{\ell_\rho(G^k)-c\|G^k\|_2^2\}\right]\le C_0 \tag{B.0.6}$$

for any m and for C_0 independent of m. The upperscript in G corresponds to the index. From [10] we know that the map $\pi_\cdot\in\mathcal{D}_{\mathcal{M}}[0,T]\mapsto\max_{k\le m}\left\{\ell_\rho(G^k)-c\|G^k\|_2^2\right\}$, is continuous and bounded with respect to the Skorohod topology. Thus, the expectation in the previous display is equal to

$$\lim_{N\to\infty}\mathbb{E}_{\mu_N}\left[\max_{k\le m}\left\{\int_0^T\frac{1}{N-1}\sum_{x=1}^{N-1}\partial_u G_s^k(\tfrac{x}{N})\eta_s(x)ds-c\|G^k\|_2^2\right\}\right]. \tag{B.0.7}$$

By entropy and Jensen's inequalities, and changing to the measure ν_γ^N plus noticing that $e^{\max_{k\le m}a_k}\le\sum_{k=1}^m e^{a_k}$, the previous display is bounded from above by

$$C_\alpha+\tfrac{1}{N-1}\log\mathbb{E}_{\nu_\gamma^N}\left[\sum_{k=1}^m e^{\int_0^T\sum_{x\in\Lambda_N}\partial_u G_s^k(\frac{x}{N})\eta_s(x)ds-cN\|G^k\|_2^2ds}\right]. \tag{B.0.8}$$

By linearity of the expectation, to treat the second term in the previous display it is enough to bound the term

$$\limsup_{N\to\infty}\ \tfrac{1}{N-1}\log\mathbb{E}_{\nu_\gamma^N}\left[e^{\int_0^T\sum_{x\in\Lambda_N}\partial_u G_s(\frac{x}{N})\eta_s(x)ds-cN\|G\|_2^2ds}\right],$$

for a fixed function $G\in C_c^{0,2}([0,T]\times(0,1))$, by a constant independent of G. By the Feynman-Kac's formula

C.4.1, the expression inside the limsup is bounded by

$$\int_0^T \sup_f \Big\{ \frac{1}{N-1} \int_{\Omega_N} \sum_{x\in\Lambda_N} \partial_u G_s(\tfrac{x}{N})\eta_s(x) f(\eta) d\nu_\gamma^N - c\|G\|_2^2 + N\langle \mathcal{L}_N\sqrt{f}, \sqrt{f}\rangle_{\nu_\gamma^N} \Big\}\, ds \tag{B.0.9}$$

where f is a density with respect to ν_γ^N. Note that by a Taylor expansion on G, we can replace ∂_u by $\nabla_N^+ G_s(\frac{x}{N})$, with an error of order $\mathcal{O}(N^{-1})$. Then, summing by parts (recall C.1.1), we get

$$\int_{\Omega_N} \sum_{x=1}^{N-2} G_s(\tfrac{x}{N})(\eta(x)-\eta(x+1)) f(\eta) d\nu_\gamma^N. \tag{B.0.10}$$

By the same trick as most of the proofs in the previous section, we write the previous term as one half of it plus one half of it, and in one of the parts we exchange $\eta \mapsto \eta^{x,x+1}$ (recall that ν_γ^N is invariant for this change of variables). Thus, the last display is equal to

$$\frac{1}{2}\int_{\Omega_N} \sum_{x=1}^{N-2} G_s(\tfrac{x}{N})(\eta(x)-\eta(x+1))(f(\eta)-f(\eta^{x,x+1})) d\nu_\gamma^N. \tag{B.0.11}$$

By analogous arguments to those used to show Lemma A.0.4, last term is bounded from above by

$$\begin{aligned} &\frac{1}{4N}\int_{\Omega_N} \sum_{x=1}^{N-2} (G_s(\tfrac{x}{N}))^2(\sqrt{f(\eta)}+\sqrt{f(\eta^{x,x+1})})^2 d\nu_\gamma^N + \frac{1}{4N}\int_{\Omega_N} \sum_{x=1}^{N-2} (\sqrt{f(\eta)}-\sqrt{f(\eta^{x,x+1})})^2 d\nu_\gamma^N \\ &\le \frac{C}{N}\sum_{x\in\Lambda_N} (G_s(\tfrac{x}{N}))^2 + \frac{1}{4N} D_{0,N}(\sqrt{f}, \nu_\gamma^N) \end{aligned} \tag{B.0.12}$$

for some $C > 0$. From (A.0.7) we get that (B.0.9) is bounded from above by

$$C' \int_0^T \Big[1 + \frac{1}{N}\sum_{x\in\Lambda_N} (G_s(\tfrac{x}{N}))^2\Big] ds \; - \; c\|G\|_2^2 \tag{B.0.13}$$

plus an error of order $O(N^{-1})$. Above C' is a positive constant independent of G.

Since we have that $\frac{1}{N}\sum_{x\in\Lambda_N}(G_s(\frac{x}{N}))^2 \xrightarrow{N\to\infty} \|G\|_2^2$,then it is enough to choose $c > C'$ to conclude that

$$\limsup_{N\to\infty} \Big\{ C' \int_0^T \Big[1 + \frac{1}{N}\sum_{x\in\Lambda_N} (G_s(\tfrac{x}{N}))^2\Big] ds \; - \; c\|G\|_2^2 \Big\} \lesssim 1 \tag{B.0.14}$$

and we are done. □

Appendix C

Auxiliary results

C.1 Discrete calculus

For any function A dependent of x, define the *backward difference operator in* x, ∇_x, as $\nabla_x A := A(x) - A(x-1)$. Then, for any A, B functions dependent of x it is easy to derive a discrete product rule.

$$\begin{aligned}\nabla_x[A(x)B(x)] &= A(x)B(x) - A(x-1)B(x-1)\\ &= (A(x) - A(x-1))B(x) + A(x-1)B(x) - A(x-1)B(x-1)\\ &= B(x)\nabla_x A(x) + A(x-1)\nabla_x B(x).\end{aligned} \tag{C.1.1}$$

By the product rule above, and considering $A(x) \equiv A_x, B(x) \equiv B_x$ one can derive the following summation by parts (also known as *Abel's lemma*, or *Abel's transformation*).

Lemma C.1.1. *Suppose* $\{A_x\}$, $\{B_x\}$ *are two sequences. Then,*

$$\sum_{x=m}^{n} A_x(B_{x+1} - B_x) = A_n B_{n-1} - A_m B_m - \sum_{x=m+1}^{n} B_x(A_x - A_{x-1}) \tag{C.1.2}$$

C.2 Dirichlet forms

The following two results can be found in [3] (Lemma 5.1 and Lemma 5.2, respectively). On the following lemma we only changed the notation to the one used through this thesis.

Lemma C.2.1. *Let* $\gamma : [0,1] \to (0,1)$ *be a function. Let* $f : \Omega_N \to \mathbb{R}^+$ *be a density with respect to the measure* $\nu^N_{\gamma(\cdot)}$. *Then, if* γ *is a constant function*

$$\langle \mathcal{L}_{N,0}\sqrt{f}, \sqrt{f}\rangle_{\nu^N_{\gamma(\cdot)}} = -D_{N,0}(\sqrt{f}, \nu^N_{\gamma(\cdot)}). \tag{C.2.1}$$

The original lemma above also has a simillar extension for γ smooth, which we will not need to use here. The following lemma is originally stated for $\alpha_1 = \alpha$ and $\gamma_1 = 1 - \alpha$. Following the proof in [3], one can easily see that the statement below holds.

Lemma C.2.2. *Let $\gamma : [0,1] \to (0,1)$ be a function. For any g density with respect to ν_γ^N we have*

$$\langle \mathcal{L}_{N,-}^L \sqrt{g}, \sqrt{g} \rangle_{\nu_{\gamma(\cdot)}^N} = -D_{N,-}^L(\sqrt{g}, \nu_{\gamma(\cdot)}^N) + \mathcal{E}_N(\gamma, g), \tag{C.2.2}$$

with $|\mathcal{E}_N(\gamma,g)| \lesssim C(\gamma)\left|\gamma(\frac{1}{N}) - \frac{\alpha_1}{\alpha_1+\gamma_1}\right| \kappa N^{-\theta} \|g\|^2_{\nu^N_{\gamma(\cdot)}}$. A similar equality holds for $\mathcal{L}_{N,b}^L$ by replacing $\left|\gamma(\frac{1}{N}) - \frac{\alpha_1}{\alpha_1+\gamma_1}\right|$ by $\left|\gamma(\frac{N-1}{N}) - \frac{\beta_1}{\beta_1+\delta_1}\right|$.

C.3 Entropy

Theorem C.3.1 (Entropy formula, [17])**.** *If μ and ν are measures in countable space E , and μ is absolutely continuous with respect to ν, the entropy $H(\mu \mid \nu)$ of μ with respect to ν is given by*

$$H(\mu \mid \nu) = \sum_{x \in E} \mu(x) \log \frac{\mu(x)}{\nu(x)}, \tag{C.3.1}$$

and is equal to ∞ otherwise.

Proposition C.3.2 (Entropy inequality)**.** *For two measures μ, ν in E and for $\gamma > 0$:*

$$\int f(\eta)\mu(d\eta) \leq \frac{1}{\gamma} H(\mu \mid \nu) + \frac{1}{\gamma} \int e^{\gamma f(\eta)} \nu(d\eta) \tag{C.3.2}$$

C.4 General inequalities

Theorem C.4.1 (Feynman-Kac inequality[3])**.** *Let $\{X_t\}_{t \geq 0}$ be a Markov process in the countable space E, with infinitesimal generator $\mathcal{L}$. Let ν be a probability measure in E and $V : [0,\infty) \times E \to \mathbb{R}$ be a bounded function. Denote $\mathcal{L}_t = \mathcal{L} + V(t)$ where $V_t = V(t, \cdot)$. Define*

$$\Gamma_t = \sup_{\|f\|_2 = 1} \left\{ \langle V_t, f^2 \rangle_\nu + \langle \mathcal{L} f, f \rangle_\nu \right\}, \tag{C.4.1}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in $L^2(E, \nu)$ and $\|\cdot\|_2 = \langle \cdot, \cdot \rangle_\nu^{\frac{1}{2}}$. Then

$$\mathbb{E}_\nu \left[e^{\int_0^t V_r(X_r) dr} \right] \leq e^{\int_0^t \Gamma_s ds}. \tag{C.4.2}$$

Theorem C.4.2 (Doob's maximal inequalities)**.** *Let $\{\mathcal{F}_t\}_{t \geq 0}$ be a filtration on a probability space $(\Omega, \mathcal{F}, P)$, and let $\{M_t\}_{t \geq 0}$ be a continuous martingale with respect to the filtration $\{\mathcal{F}_t\}_{t \geq 0}$.*

- *Let $p \geq 1$ and $T > 0$. If $\mathbb{E}\left[|M_T|^p\right] < +\infty$ then $\forall t > 0$,*

$$\mathbb{P}(\sup_{0 \leq t \leq T} |M_t| \geq \lambda) \leq \lambda^{-p} \mathbb{E}\left[|M_T|^p\right]. \tag{C.4.3}$$

- *Let $p > 1$ and $T > 0$. If $\mathbb{E}\left[|M_T|^p\right] < +\infty$ then*

$$\mathbb{E}[(\sup_{0 \leq t \leq T} |M_t|)^p] \leq \left(\frac{p}{p-1}\right)^p \mathbb{E}\left[|M_T|^p\right]. \tag{C.4.4}$$

Doob's inequality is a classical result in Stochastic Calculus and both the proof and statement can be found in most books on the subject.

Theorem C.4.3 (Markov's inequality)**.** *Let X be a r.v. such that $\mathbb{E}\left[|X|^p\right] < \infty$. Then $\forall \lambda > 0$,*

$$\mathbb{P}(|X| \geq \lambda) \leq \lambda^{-p}\mathbb{E}\left[|X|^p\right]. \tag{C.4.5}$$

C.5 Topology

The following results can be found in [17], chapter 4

C.5.1 Tightness

Definition C.5.1. A collection of probability measures $\mathcal{M}$ defined in a metric space $(\mathcal{S}, \mathcal{F})$ is *tight* if $\forall \epsilon > 0$ exist K compact such that $\forall \mu \in \mathcal{M}$ we have $\mu(K) > 1 - \epsilon$.

Definition C.5.2 (Relatively compact)**.** A collection of probability measures $\mathcal{M}$ is relatively compact if each sequence has a subsequence that converges (weakly).

Theorem C.5.3 (Prokhorov's theorem)**.** *Let $\mathcal{M}$ be a collection of probability measures on $(\mathcal{S}, \mathcal{F})$. If $\mathcal{M}$ is tight, then $\mathcal{M}$ is relatively compact. Moreover, if $\mathcal{S}$ is complete and separable, the reciprocal is also true.*

C.5.2 Skorohod topology

Let $\mathcal{D}_{\mathcal{S}}[0, T]$ be the space of *càdlàg* functions with values in $\mathcal{S}$, where $\mathcal{S}$ is a separable space endowed with a distance δ.

In order to avoid big oscilations in the measure as a result of the jumps of the process, we use a more "smooth" distance than the uniform – the *Skorohod measure*. First, let us define $\Lambda = \{\lambda \mid [0, T] \to [0, T] : \lambda \quad \text{is a continuous and increasing function}\}$. If $\lambda \in \Lambda$ we let

$$\|\lambda\| := \sup_{t \neq s} \left| \log\left(\frac{\lambda(t) - \lambda(s)}{t - s} \right) \right|. \tag{C.5.1}$$

Finally, if $\mu, \nu \in \mathcal{S}$ we define the *Skorohod distance* as

$$d(\mu, \nu) = \inf_{\lambda \in \Lambda} \max\{\|\lambda\|, \sup_{0 \leq t \leq T} \delta(\mu_t, \nu_{\lambda(t)})\}. \tag{C.5.2}$$

Proposition C.5.4. *The space $\mathcal{D}_{\mathcal{S}}[0, T]$ endowed with the metric d above is a metric complete separable space.*

The *module of uniform continuity* of a trajectory μ is defined as

$$\omega_\mu(\gamma) = \sup_{|s-t| \leq \gamma} \delta(\mu_s, \mu_t). \tag{C.5.3}$$

We can modify the definition above to a more useful one:

$$\omega'_{\mu}(\gamma) := \inf_{\{t_i\}_{0\le i<r}} \max_{0\le i<r} \sup_{t_i\le s<t<t_{i+1}} \delta(\mu_s, \mu_t), \tag{C.5.4}$$

where the infimum on $\{t_i\}_{0\le i<r}$ is taken over all the partitions of $[0,T]$ such that $0 =: t_0 < \cdots < t_r := T$, and $t_i - t_{i-1} > \gamma, \forall 1 \le i \le r$.

Theorem C.5.5 (Aldous' conditions)**.** *Let $\{P^N\}_{N\ge1}$ be a sequence of probability measures in $\mathcal{D}_{\mathcal{S}}[0,T]$. Then the sequence is relatively compact if and only if*

1. *For all $t \in [0,T], \epsilon > 0$ there exists $K_t(\epsilon) \subset \mathcal{S}$ compact such that*

$$\sup_{N\ge1} P^N\left(\mu \in \mathcal{D}_{\mathcal{S}}[0,T] : \mu_t \notin K_t(\epsilon)\right) < \epsilon. \tag{C.5.5}$$

2. $\forall \epsilon > 0$

$$\lim_{\gamma\to0} \limsup_{N\to\infty} P^N\left(\mu \in \mathcal{D}_{\mathcal{S}}[0,T] : \omega'_{\mu}(\gamma) > \epsilon\right) = 0. \tag{C.5.6}$$

Condition 2. of the previous theorem can be replaced by a more simple to check:

$$\lim_{\gamma\to0} \limsup_{N\to\infty} P^N\left(\mu \in \mathcal{D}_{\mathcal{S}}[0,T] : \omega_{\mu}(\gamma) > \epsilon\right) = 0. \tag{C.5.7}$$

The result that we will apply is the following which guarantees the requirement of (C.5.7)

Proposition C.5.6. *A sequence of probability measures $\{P^N\}_{N\ge1}$ in $\mathcal{D}_{\mathcal{S}}[0,T]$ satisfies (C.5.7) if $\forall \epsilon > 0$,*

$$\lim_{\gamma\to0} \limsup_{N\to\infty} \sup_{\tau\in\mathcal{T}_T, \theta\le\gamma} P^N\left(\mu \in \mathcal{D}_{\mathcal{S}}[0,T] : \delta(\mu_{\tau+\theta,\mu_\tau}) > \epsilon\right) = 0, \tag{C.5.8}$$

where $\mathcal{T}_T$ is the set of all stopping times bounded (a.s.) by T.

Proposition C.5.7. *Let $\{g_k\}_{k\ge1}$ be a dense family in $C^2_0[0,1]$. A sequence of probability measures $\{\mathbb{Q}^N\}_{N\ge1} \in \mathcal{D}_{\mathcal{M}}[0,T]$ is relatively compact if $\forall k \ge 0 \quad \{\mathbb{Q}^{N,g_k}\}_{N\ge1}$ is, where $\mathbb{Q}^{N,g_k}$ is induced by the map*

$$\phi : \left(\mathcal{D}_{\mathcal{M}}[0,T], \mathbb{Q}^N\right) \longrightarrow \left(\mathcal{D}_{\mathbb{R}}[0,T], \mathbb{Q}^{N,g_k}\right) \tag{C.5.9}$$

$$\{\pi^N_t\}_{t\ge0} \mapsto \{\langle \pi^N_t, g_k\rangle\}_{t\ge0}. \tag{C.5.10}$$

The result above is consequence of the duality $\langle \pi^N_t, g_k\rangle$. Now we state *Portmanteau's theorem*, which is a classical result in probability theory and we refer the reader to [16] for more details.

Theorem C.5.8 (Portmanteau's theorem)**.** *For any random elements $X, X_1, X_2, \ldots$ in a metric space $\mathcal{S}$, the following conditions are equivalent*

1. $X_n \xrightarrow{d} X$.

2. $\liminf_N P(X_N \in G) \ge P(X \in G)$ *for any open set $G \subset \mathcal{S}$.*

3. $\limsup_N P(X_N \in F) \leq P(X \in F)$ *for any closed set* $F \subset \mathcal{S}$.

4. $P(X_N \in B) \to P(X \in B)$ *for any* $B \in \mathcal{B}(S)$ *with* $X \notin \partial B$ *a.s.,*

where $\mathcal{B}(S)$ *denotes the Borel* $\sigma-$*algebra in* $\mathcal{S}$*, and* ∂B *the boundary of* B.

C.6 Weak solutions

In this section we will define our notion of weak solution of the heat equation with both Robin and Neumann boundary conditions. In this way, consider the heat equation

$$\begin{cases} \partial_t \rho_t(u) = \triangle \rho_t(u), & (t,u) \in [0,T] \times (0,1) \\ \partial_u \rho_t(0) = u_0, & t \in [0,T] \\ \partial_u \rho_t(1) = u_1, & t \in [0,T]. \end{cases} \tag{C.6.1}$$

First we will (informally) derive a weak solution to the equation above, then we will define the boundary conditions. Multiplying a test function $H \in C^{1,2}([0,T] \times [0,1])$ in $\partial_t \rho = \triangle \rho$ we get:

$$\partial_t \rho_t(u) H_t(u) = \triangle \rho_t(u) H_t(u). \tag{C.6.2}$$

Integrating on $[0,T] \times (0,1)$:

$$I_1 := \int_0^1 \int_0^T \partial_t \rho_t(u) H_t(u) dt du = \int_0^1 \int_0^T \triangle \rho_t(u) H_t(u) dt du =: I_2. \tag{C.6.3}$$

Integrating by parts I_1:

$$I_1 = \int_0^1 \left(\rho_T(u) H_T(u) - \rho_0(u) H_0(u) \right) du - \int_0^1 \int_0^T \rho_t(u) \partial_t H_t(u) dt du. \tag{C.6.4}$$

Integrating I_2 by parts twice:

$$\begin{aligned} I_2 &= \int_0^T \partial_u \rho_t(1) H_t(1) - \partial_u \rho_t(0) H_t(0) dt - \int_0^T \int_0^1 \partial_u \rho_t(u) \partial_u H_t(u) du dt \\ &= \int_0^T \partial_u \rho_t(1) H_t(1) - \partial_u \rho_t(0) H_t(0) dt - \int_0^T \rho_t(1) \partial_u H_t(1) - \rho_t(0) \partial_u H_t(0) dt \\ &+ \int_0^T \int_0^1 \rho_t(u) \triangle H_t(u) du dt. \end{aligned} \tag{C.6.5}$$

Equating I_1 and I_2 and using Fubini's theorem:

$$\begin{aligned} &\int_0^1 \left(\rho_T(u) H_T(u) - \rho_0(u) H_0(u) \right) du - \int_0^1 \int_0^T \rho_t(u) \partial_t H_t(u) dt du \\ &= \int_0^T \partial_u \rho_t(1) H_t(1) - \partial_u \rho_t(0) H_t(0) dt - \int_0^T \rho_t(1) \partial_u H_t(1) - \rho_t(0) \partial_u H_t(0) dt \\ &+ \int_0^T \int_0^1 \rho_t(u) \triangle H_t(u) du dt, \end{aligned} \tag{C.6.6}$$

that is:

$$
\begin{aligned}
0 = &\int_0^1 \left(\rho_T(u)H_T(u) - \rho_0(u)H_0(u)\right) du - \int_0^T \int_0^1 \rho_t(u)(\partial_t + \triangle)H_t(u)dudt - \\
&- \int_0^T \partial_u\rho_t(1)H_t(1) - \partial_u\rho_t(0)H_t(0)dt + \int_0^T \rho_t(1)\partial_u H_t(1) - \rho_t(0)\partial_u H_t(0)dt
\end{aligned} \tag{C.6.7}
$$

Definition C.6.1. A function $\rho : [0,T] \times [0,1] \longrightarrow [0,1]$ is a weak solution of the heat equation (C.6.1) with *Robin boundary conditions* if ρ satisfies the weak formulation (C.6.7) with

$$
\begin{aligned}
u_0 &= -\kappa(\alpha_1 - (\alpha_1 + \gamma_1)\rho_t(1) + (\gamma_2 - \alpha_2)(\rho_s^2(1) - \rho_t(1))) \\
u_1 &= \kappa(\beta_1 - (\beta_1 + \delta_1)\rho_t(1) + (\delta_2 - \beta_2)(\rho_s^2(1) - \rho_t(1)))
\end{aligned} \tag{C.6.8}
$$

for any function $H \in C^2([0,T] \times [0,1])$, and $\rho \in L^2(0,T;\mathcal{H}^1(0,1))$.

If, instead, we have ρ satisfies the weak formulation (C.6.7) for any function $H \in C^2([0,T] \times [0,1])$, and $\rho \in L^2(0,T;\mathcal{H}^1(0,1))$ but

$$
u_0 = u_1 = 0, \tag{C.6.9}
$$

we say that ρ is a weak solution to the heat equation with *Neumann boundary conditions*.

Explicitly, these formulations take the form:

$$
\begin{aligned}
F_R := &\int_0^1 \left(\rho_T(u)H_T(u) - \rho_0(u)H_0(u)\right) du - \int_0^T \int_0^1 \rho_t(u)(\partial_t + \triangle)H_t(u)dudt - \\
&- \int_0^T H_t(1)(\kappa(\beta_1 - (\beta_1 + \delta_1)\rho_t(1) + (\delta_2 - \beta_2)(\rho_s^2(1) - \rho_t(1))))dt - \\
&- \int_0^T H_t(0)(\kappa(\alpha_1 - (\alpha_1 + \gamma_1)\rho_t(1) + (\gamma_2 - \alpha_2)(\rho_s^2(1) - \rho_t(1))))dt + \\
&+ \int_0^T \rho_t(1)\partial_u H_t(1) - \rho_t(0)\partial_u H_t(0)dt = 0,
\end{aligned} \tag{C.6.10}
$$

and

$$
\begin{aligned}
F_N := &\int_0^1 \left(\rho_T(u)H_T(u) - \rho_0(u)H_0(u)\right) du - \int_0^T \int_0^1 \rho_t(u)(\partial_t + \triangle)H_t(u)dudt + \\
&+ \int_0^T \rho_t(1)\partial_u H_t(1) - \rho_t(0)\partial_u H_t(0)dt = 0.
\end{aligned} \tag{C.6.11}
$$

Appendix D

Matrix Product Ansatz

D.1 Coefficients' roots

Let

$$
\begin{aligned}
z &= (\alpha_1+\gamma_1)(\alpha_1(\alpha_2+\gamma_3)+(\alpha_3+\gamma_2)(\alpha_2+\gamma_1+\gamma_3))+ \\
&+ N^\theta((\alpha_1+\alpha_2)(\alpha_1+\alpha_3)+(\gamma_1+\gamma_3)(\gamma_1+\gamma_2)+2(\gamma_1+\gamma_3)(\alpha_1+\alpha_3))
\end{aligned}
$$

then

$$
\begin{aligned}
d_1^L z &= N^\theta(\alpha_1+\alpha_3)(\alpha_1+\alpha_2+\gamma_1+\gamma_3)+ \\
&\quad + \alpha_1(\alpha_1(\alpha_2+\gamma_3)+(\alpha_3+\gamma_2)(\alpha_2+\gamma_1+\gamma_3)) \\
d_2^L z &= N^{2\theta}(\alpha_2-\alpha_3-\gamma_2+\gamma_3) \\
d_3^L z &= -N^{2\theta}(\alpha_1+\alpha_2+\gamma_1+\gamma_3) \\
t_1^L z &= \alpha_1\gamma_1(\gamma_2-\alpha_2)+\alpha_1\gamma_3(\alpha_3+\gamma_2)-\alpha_3\gamma_1(\alpha_2+\gamma_3)+\alpha_1^2\gamma_3-\alpha_3\gamma_1^2 \\
t_2^L z &= -N^\theta(\alpha_1+\gamma_1)(\alpha_2-\alpha_3-\gamma_2+\gamma_3) \\
t_3^L z &= N^\theta(\alpha_1+\gamma_1)(\alpha_1+\alpha_2+\gamma_1+\gamma_3) \\
f_1^L z &= N^\theta(\alpha_1+\alpha_2)(\alpha_1+\alpha_3)+\alpha_1(\alpha_1\alpha_2+\alpha_3\gamma_1+\alpha_2(\alpha_3+\gamma_2)) \\
f_2^L z &= -N^\theta((\alpha_1+\gamma_1)(\alpha_3+\gamma_2)+N^\theta(\alpha_1-\alpha_2+2\alpha_3+\gamma_1+\gamma_2) \\
f_3^L z &= -N^\theta(\alpha_1(\alpha_1+\gamma_1)+N^\theta(\alpha_1+\alpha_2))
\end{aligned}
$$

and for the right

$$
\begin{aligned}
d_1^R z &= N^\theta(\beta_1+\beta_3)(\beta_1+\beta_2+\delta_1+\delta_3)+ \\
&\quad + \beta_1(\beta_1(\beta_2+\delta_3)+(\beta_3+\delta_2)(\beta_2+\delta_1+\delta_3)) \\
d_2^R z &= -N^{2\theta}(\beta_2-\beta_3-\delta_2+\delta_3) \\
d_3^R z &= N^{2\theta}(\beta_1+\beta_2+\delta_1+\delta_3)
\end{aligned}
$$

$$t_1^R z = -(\beta_1\delta_1(\delta_2-\beta_2)+\beta_1\delta_3(\beta_3+\delta_2)-\beta_3\delta_1(\beta_2+\delta_3)+\beta_1^2\delta_3-\beta_3\delta_1^2)$$
$$t_2^R z = -N^\theta(\beta_1+\delta_1)(\beta_2-\beta_3-\delta_2+\delta_3)$$
$$t_3^R z = N^\theta(\beta_1+\delta_1)(\beta_1+\beta_2+\delta_1+\delta_3)$$
$$f_1^R z = N^\theta(\beta_1+\beta_2)(\beta_1+\beta_3)+\beta_1(\beta_1\beta_2+\beta_3\delta_1+\beta_2(\beta_3+\delta_2))$$
$$f_2^R z = N^\theta((\beta_1+\delta_1)(\beta_3+\delta_2)+N^\theta(\beta_1-\beta_2+2\beta_3+\delta_1+\delta_2)$$
$$f_3^R z = N^\theta(\beta_1(\beta_1+\delta_1)+N^\theta(\beta_1+\beta_2))$$

We can relate the left and right coefficients more elegantly as:

$$d_1^L(\alpha,\gamma)=d_1^R(\alpha,\gamma) \qquad d_2^L(\alpha,\gamma)=-d_2^R(\alpha,\gamma) \qquad d_3^L(\alpha,\gamma)=-d_3^R(\alpha,\gamma)$$
$$t_1^L(\alpha,\gamma)=-t_1^R(\alpha,\gamma) \qquad t_2^L(\alpha,\gamma)=t_2^R(\alpha,\gamma) \qquad t_3^L(\alpha,\gamma)=-t_3^R(\alpha,\gamma)$$
$$f_1^L(\alpha,\gamma)=f_1^R(\alpha,\gamma) \qquad f_2^L(\alpha,\gamma)=-f_2^R(\alpha,\gamma) \qquad f_3^L(\alpha,\gamma)=-f_3^R(\alpha,\gamma)$$

D.2 Rates matrices

$$b_L^* = \begin{pmatrix} -(\alpha_1+\alpha_3) & 2\alpha_1+2\alpha_3+\gamma_1+\gamma_2 & -(\alpha_1+\alpha_3+\gamma_1+\gamma_2) & -(\alpha_1+\alpha_3+\gamma_2) \\ \alpha_3 & -(\alpha_1+2\alpha_3+\gamma_2) & \alpha_1+\alpha_3+\gamma_1+\gamma_2 & \alpha_1+\alpha_3+\gamma_2 \\ \alpha_1 & -(2\alpha_1+\alpha_2+\gamma_1) & \alpha_1+\alpha_2+\gamma_1+\gamma_3 & \alpha_1 \\ 0 & \alpha_1+\alpha_2 & -(\alpha_1+a_2+\gamma_1+\gamma_3) & -\alpha_1 \end{pmatrix} \tag{D.2.1}$$

$$b_R^* = \begin{pmatrix} -(\beta_1+\beta_3) & 2\beta_1+2\beta_3+\delta_1+\delta_2 & -(\beta_1+\beta_3+\delta_1+\delta_2) & \beta_1+\beta_3+\delta_2 \\ \beta_1 & -(2\beta_1+\beta_2+\delta_1) & \beta_1+\beta_2+\delta_1+\delta_3 & -\beta_1 \\ \beta_3 & -(\beta_1+2\beta_3+\delta_2) & \beta_1+\beta_3+\delta_1+\delta_2 & -(\beta_1+\beta_3+\delta_2) \\ 0 & \beta_1+\beta_2 & -(\beta_1+\beta_2+\delta_1+\delta_3) & \beta_1 \end{pmatrix} \tag{D.2.2}$$

$$m_L^* = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{D.2.3}$$

Appendix E

Auxiliary results for the propagation of chaos

E.1 Bounds for a single random walk with reflections

As in [24], we state some properties of a single random walk in Λ_N with reflections at the boundaries. For the proof of the following results we refer the reader to [25]. Although in [24], the jump rates of the SSEP are $1/2$, instead of 1, as through this thesis, the 2 in the denominator can be incorporated by the constants in the bounds.

Let $P_t^{(\epsilon)}(x,y)$ be the transition probability of a simple random walk in Λ_N which jumps with intensity $\epsilon^{-2}/2$ to its nearest neighbor sites. As in Section 4.1, jumps to outside of Λ_N are supressed. Define the quantity

$$G_t(x,y) = \frac{e^{-\frac{x-y}{2t}}}{\sqrt{2\pi t}}, \tag{E.1.1}$$

also known as *Gauss kernel*. Then one can show that we have

$$P_t^{(\epsilon)}(x,y) \leq cG_{\epsilon^{-2}t}(x,y), \quad \forall \epsilon > 0, t \in [0,T], x,y \in \Lambda_N \tag{E.1.2}$$

and

$$\left| P_t^{(\epsilon)}(x,y) - P_t^{(\epsilon)}(x+1,y) \right| \leq \frac{c}{\sqrt{\epsilon^{-2}t}} G_{\epsilon^{-2}t}(x,y), \forall \epsilon > 0, t \in [0,T], 1 \leq x \leq N-1. \tag{E.1.3}$$

In Proposition 5.1 of [25] it is shown some bounds for the gradients $|\rho_\epsilon(x,t) - \rho_\epsilon(x+1,t)|$. Using the bounds shown there, one can get the following bound $\forall \xi > 0$ and $\tau > 0$

$$\sup_{x,y\in\Lambda_N:|x-y|\leq 1} |\rho_\epsilon(x,t) - \rho_\epsilon(y,t)| \leq \frac{c}{(\epsilon^{-2}t)^{1/2-\xi}+1}, \quad \forall t \leq \tau \log \epsilon^{-1}. \tag{E.1.4}$$

E.2 Inequalities

The proof for the following inequality can be found in [1]

Theorem E.2.1 (Andjel)**.** *If A and B are disjoint subsets of S, then for all $\eta \in \{0,1\}^{\Lambda_N}$ and $t \geq 0$,*

$$P(\prod_{x\in A\cup B}\eta(t,x)=1) \leq P(\prod_{x\in A}\eta(t,x)=1)P(\prod_{x\in B}\eta(t,x)=1) \tag{E.2.1}$$