Exercise list number 1 - σ-algebras and probability measures

Exercise 1:

Show that, if $\mathscr{A}$ and $\mathscr{B}$ are two σ-algebras, then $\mathscr{A}\bigcap\mathscr{B}$ is also a σ-algebra.

Exercise 2:

Let $\Omega := \{\omega_1, \omega_2, \omega_3\}$ be a sample space.

1. Exhibit all the σ-algebras of Ω.
2. Compute $\sigma(\{\omega_1\})$. Check that it is a σ-algebra.

Exercise 3:

Recall that, for a topological space S the Borel σ-algebra $\mathscr{B}(S)$ is generated by the family of open subsets of S. Prove that the Borel σ-algebra of $\mathbb{R}$ is generated by $\pi(\mathbb{R}) = \{(-\infty, x] : x \in \mathbb{R}\}$.

Exercise 4:

Let X be a random variable defined on a sample space Ω. Compute $\sigma(X)$, that is the σ-algebra generated by X, when

1. $\Omega := \{\omega_1, \omega_2, \omega_3\}$ and $X(\omega_1) = X(\omega_2) = X(\omega_3) = 1$.
2. $\Omega := \{\omega_1, \omega_2, \omega_3\}$ and $X(\omega_1) = 0$, $X(\omega_2) = 1$ and $X(\omega_3) = 2$.
3. $\Omega := \{\omega_1, \omega_2, \omega_3\}$ and $X(\omega_1) = 0$, $X(\omega_2) = 0$ and $X(\omega_3) = 1$.
4. $\Omega := \{\omega_1, \omega_2, \omega_3, \omega_4\}$ and $X(\omega_1) = 0$, $X(\omega_2) = 0$, $X(\omega_3) = 1$ and $X(\omega_4) = 2$.

Exercise 5:

Let Ω be a sample space, $\mathscr{F}$ be a σ-algebra of subsets of Ω.
Assume that $\mu(\cdot)$ is a set map defined on Ω satisfying the following conditions:

1. $\forall E \in \mathscr{F}$, $\mu(E) \geq 0$;
2. If $\{E_j\}_{j\geq 1}$ is a countable collection of disjoint sets in $\mathscr{F}$, then
$$\mu\Big(\bigcup_{j\geq 1} E_j\Big) = \sum_{j\geq 1} \mu(E_j);$$
3. $\mu(\Omega) = 1$.

Prove that

1. $\forall E \in \mathscr{F}$, $\mu(E) \leq 1$;
2. $\forall E \in \mathscr{F}$, $\mu(\emptyset) = 0$;
3. $\forall E \in \mathscr{F}$, $\mu(E) = 1 - \mu(E^c)$;
4. $\forall E, F \in \mathscr{F}$, $\mu(E\bigcup F) + \mu(E\bigcap F) = \mu(E) + \mu(F)$;

5. $\forall E, F \in \mathscr{F}$ such that $E \subseteq F$, $\mu(E) = \mu(F) - \mu(F \setminus E) \leq \mu(F)$;

6. Let $\{E_j\}_{j\geq 1}$ be an increasing (decreasing) sequence of sets in $\mathscr{F}$ that is $E_j \subseteq E_{j+1}$ ($E_j \supseteq E_{j+1}$) for all $j \geq 1$. Prove that, if $\{E_j\}_{j\geq 1}$ is an increasing (decreasing) sequence of sets in $\mathscr{F}$ such that $E_j \uparrow E$ ($E_j \downarrow E$), that is $E = \bigcup_{j\geq 1} E_j$ ($E = \bigcap_{j\geq 1} E_j$), then $\lim_{j\to+\infty} \mu(E_j) = \mu(E)$;

7. (Boole's inequality): $\mu\Big(\bigcup_{j\geq 1} E_j\Big) \leq \sum_{j\geq 1} \mu(E_j)$.

Exercise 6:

Let $\{E_j\}_{j\geq 1}$ be random events belonging to $\mathscr{F}$, a σ-field of events of a sample space Ω.
Let $\mu(\cdot)$ be a probability measure defined on $\mathscr{F}$. Show that for all $n \geq 1$

1. $\mu\Big(\bigcap_{j=1}^n E_j\Big) \geq 1 - \sum_{j=1}^n \mu(E_j^c)$;

2. If $\mu(E_j) \geq 1 - \varepsilon$, for $j \in \{1, \cdots, n\}$, then $\mu\Big(\bigcap_{j=1}^n E_j\Big) \geq 1 - n\varepsilon$;

3. $\mu\Big(\bigcap_{j\geq 1} E_j\Big) \geq 1 - \sum_{j\geq 1} \mu(E_j^c)$;

Exercise 7:

Prove the following properties:

1. If $\mu(E_j) = 0$ for all $j \geq 1$, then $\mu\Big(\bigcup_{j\geq 1} E_j\Big) = 0$;

2. If $\mu(E_j) = 1$ for all $j \geq 1$, then $\mu\Big(\bigcap_{j\geq 1} E_j\Big) = 1$;

Exercise 8:

Take $\{E_j\}_{j\geq 1}$ and $\{F_j\}_{j\geq 1}$ belonging to the same probability space $(\Omega, \mathscr{F}, \mu)$.
Suppose that $\lim_{j\to+\infty} \mu(E_j) = 1$ and $\lim_{j\to+\infty} \mu(F_j) = p$, with $p \in [0,1]$.
Show that $\lim_{j\to+\infty} \mu(E_j \bigcap F_j) = p$.

Exercise 9:

Let

$$\limsup_n E_n = \bigcap_{n\geq 1} \bigcup_{k\geq n} E_k, \tag{1}$$

$$\liminf_n E_n = \bigcup_{n\geq 1} \bigcap_{k\geq n} E_k. \tag{2}$$

If (2) and (1) are equal we write

$$\lim_n E_n = \liminf_n E_n = \limsup_n E_n.$$

Let $\{E_n\}_{n\geq 1}$ belong to a probability space $(\Omega, \mathscr{F}, \mu)$. Show that

1. $$\mu\Big(\liminf_n E_n\Big) \leq \liminf_n \mu(E_n) \leq \limsup_n \mu(E_n) \leq \mu\Big(\limsup_n E_n\Big).$$

2. If $\lim_{n\to+\infty} E_n = E$, then $\lim_{n\to+\infty} \mu(E_n) = \mu(E)$.

Exercise list number 2 - Random variables and distribution functions

Exercise 1:

Specify the distribution function and the distribution measure of the random variable X.

(a) If X has probability function defined on $k \in \{0,1\}$ and given by

$$\mathbb{P}(X=k)=p^k(1-p)^{1-k}.$$

That is X has Bernoulli distribution of parameter p.

(b) If X has probability function defined in $k \in \{0,\cdots,n\}$ and given by

$$\mathbb{P}(X=k)=C_k^n p^k(1-p)^{n-k}.$$

That is X has Binomial distribution of parameter n and p.

(c) If X has probability function defined in $k \in \{0,1,\cdots\}$ and given by

$$\mathbb{P}(X=k)=\frac{e^{-\alpha}\alpha^k}{k!},$$

$\alpha>0$. That is X has Poisson distribution of parameter α.

(d) If X has probability function defined in $k \in \{0,1,\cdots\}$ and given by

$$\mathbb{P}(X=k)=p(1-p)^k.$$

That is X has Geometric distribution of parameter p.

(e) If X has probability density function given by

$$f(x)=\alpha e^{-\alpha x}\mathbf{1}_{[0,+\infty)}(x),$$

with $\alpha>0$. That is X has Exponential distribution with parameter α.

(f) If X has probability density function given by

$$f(x)=\frac{1}{b-a}\mathbf{1}_{[a,b]}(x)$$

for $a,b \in \mathbb{R}$ with $a<b$. That is X has Uniform distribution in $[a,b]$.

(g) If X has probability density function given by

$$f(x)=\frac{1}{\pi(1+x^2)},$$

$x \in \mathbb{R}$. That is X has Cauchy distribution.

(h) If X has probability density function given by

$$f(x)=\frac{1}{\sqrt{2\pi}}e^{-x^2/2},$$

$x \in \mathbb{R}$. That is X has Gaussian distribution.

Exercise 2:

Let $\sigma > 0$. Let X be a r.v. with probability density function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}$.

(a) Prove that $f(\cdot)$ is indeed a probability density function. How does the graph of f look like when σ is very small?

(b) Compute $\mathbb{E}[X]$ and $\mathbb{E}[X^2]$.

Exercise 3:

Let X be a random variable with probability density function given by $f(x) = cx^2\mathbf{1}_{[-1,1]}(x)$.

(a) Determine the value of the constant c.

(b) Exhibit the distribution function $F_X(\cdot)$ and find x_1 such that $F_X(x_1) = 1/4$.

Exercise 4:

Let X be a random variable with distribution function given by $F_X(x) = x^3\mathbf{1}_{[0,1]}(x) + \mathbf{1}_{(1,\infty]}(x)$.

(a) Find the probability density function of X.

(b) Prove that it is indeed a probability density function.

Exercise 5:

A random variable X is said to be symmetric around μ if $\mathbb{P}(X \geq \mu + x) = \mathbb{P}(X \leq \mu - x)$ for all $x \in \mathbb{R}$. If $\mu = 0$ we simply say that X is symmetric.
Let X be a random variable symmetric around the point $b \in \mathbb{R}$ and suppose that X takes the values a, b and $2b - a$, with $a < 0$ and $b > 0$.

(a) Show that $\mathbb{E}[X] = b$.

(b) Suppose that $\mathbb{E}[X] = 1$, $a = -1$, $Var(X) = 3$ and determine the distribution function of X and its induced measure μ_X.

(c) Compute $\mu_X((-\infty, -1])$, $\mu_X((-\infty, 3/2])$ and $\mu_X(\{1\})$.

Exercise 6:

Let X be a symmetric random variable that takes the values $a \neq b \neq c$.
Suppose that $\mathbb{P}(X = 0) = 1/5$.
Give the results in terms of $a \neq 0$.

(a) Exhibit the distribution function and the distribution measure of X.

(b) Compute $\mathbb{E}[X]$ and $Var(X)$.

Exercise 7:

Let X be a random variable with probability density function $f_X(\cdot)$ and for $b > 0$ and $c \in \mathbb{R}$ let $Y = bX + c$.

(a) Prove that the probability density function of Y is given by $f_Y(y) = \frac{1}{b} f_X\left(\frac{y-c}{b}\right)$.

(b) Let X be a random variable with Cauchy distribution.
Compute the probability density function of $Y = bX + M$, where $b > 0$ and $M \in \mathbb{R}$.

(c) Let X be a random variable with standard Normal distribution.
Compute the probability density function of $Y = \sigma X + \mu$, where $\sigma > 0$ and $\mu \in \mathbb{R}$.

(d) Let X be a random variable with Gamma distribution with parameter α and 1.

Compute the probability density function of $Y = \frac{X}{\beta}$.
What is the distribution of Y when $\alpha = 1$?

Exercise 8:

Let X be a random variable with density function given by $f(x) = (1+x)^{-2}\mathbf{1}_{(0,+\infty)}(x)$.
Let $Y = \max(X, c)$, where c is a positive constant $c > 0$.

(a) Show that $f(\cdot)$ is a probability density function.

(b) Exhibit the distribution function of X and Y. Justify that F_X is in fact a distribution function.

(c) Decompose $F_Y(\cdot)$ in its discrete, absolutely continuous and singular parts.

(d) Compute $\mathbb{E}[X]$ and $\mathbb{E}[Y]$.

Exercise 9:

Let X be a random variable uniformly distributed on the interval $[0, 1]$.
Let Y be the random variable defined as $Y = min(1/2, X)$.

(a) Determine the distribution function of X and Y and represent their graph.

(b) Decompose $F_Y(\cdot)$ in its discrete, absolutely continuous and singular parts.

(c) Compute $\mathbb{E}[X]$ and $\mathbb{E}[Y]$.

Exercise 10:

Let X be a random variable with exponential distribution with parameter $\lambda > 0$. Let $Y = \max(X, \lambda)$.

(a) Determine the distribution function of X and Y and represent their graph.

(b) Decompose $F_Y(\cdot)$ in its discrete, absolutely continuous and singular parts.

Exercise 11:

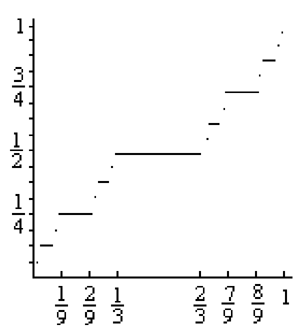
Let X be a random variable uniformly distributed on $[0, 2]$.
Let Y be the random variable defined by $Y = min(1, X)$.

(a) Determine the distribution functions of X and Y and represent their graph.

(b) Decompose $F_Y(\cdot)$ in its discrete, absolutely continuous and singular parts.

Exercise 12:

Let X be a random variable with Cantor distribution:



(a) Describe the construction of its distribution function $F_X(\cdot)$.

(b) Justify that X is a singular random variable.

(c) Compute $\mathbb{P}\left(X = \frac{1}{3}\right)$. Justify.

(d) Compute $\mathbb{P}\left(\frac{1}{3} < X < \frac{2}{3}\right)$, $\mathbb{P}(X \leq \frac{2}{3})$ and $\mathbb{P}(\frac{1}{9} < X \leq \frac{8}{9})$.

(e) Compute $\mathbb{E}[X]$. Justify.

Exercise 13: Let U be a random variable uniformly distributed in $[0,1]$.

(a) Find a function $f : [0,1] \to \mathbb{R}$ such that $f(U)$ is a random variable uniform in $[0,2]$.

(b) Find a function $f : [0,1] \to \mathbb{R}$ such that $f(U)$ is a random variable with Bernoulli distribution of parameter p, where $p \in (0,1)$.

(c) Find a function $f : [0,1] \to \mathbb{R}$ such that $f(U)$ is a random variable with exponential distribution of parameter $\lambda > 0$.

(d) Let $0 < p < q < 1$. Construct a random vector (X,Y) such that X has distribution Bernoulli with parameter p, Y has distribution Bernoulli with parameter q and $X \leq Y$ almost surely.

(e) Let $0 < \lambda_1 < \lambda_2$. Construct a random vector (X,Y) such that X has exponential distribution with parameter λ_1, Y has exponential distribution with parameter λ_2 and $X \geq Y$ almost surely.

Exercise list number 3 - Random vectores. Stochastic Independence.

Exercise 1:

Select a point uniformly in the unitary circle $\mathscr{C} = \{(x,y): x^2+y^2 \leq 1\}$.
Let X and Y be the coordinates of the selected point.

(a) Determine the joint density of X and Y.

(b) Determine $\mathbb{P}(X<Y)$, $\mathbb{P}(X>Y)$ and $\mathbb{P}(X=Y)$.

(c) What is probability of finding the point in the first quadrant? Justify.

Exercise 2:

Suppose that X and Y are random variables identically distributed with symmetric distribution around zero and with joint distribution given by

$X \setminus Y$	-1	0	...
-1	...	0	...
0	0	...	0
...	θ	0	θ

(a) If $\mathbb{P}(X=-1)=2/5$, complete the table.

(b) Compute $\mathbb{E}[X]$, $\mathbb{E}[Y]$ and $Var(X)$.

(c) Are the random variables X and Y independent? Justify.

(d) Find the probability functions of the random variables $X+Y$ and XY.
Justify if $X+Y$ and XY are symmetric random variables around zero.

(e) Represent the graph of the distribution function of the random variable $X+Y$.

(f) Explicit the measure μ_{X+Y}.

(g) Compute $\mu_{X+Y}(\{0\})$ and $\mu_{X+Y}((-\infty,0])$.

Exercise 3:

Suppose that X and Y are random variables with joint distribution given by:

$X \setminus Y$	1	2	3
1	0	1/5	0
2	1/5	1/5	1/5
3	0	1/5	0

(a) Compute the marginal probability functions of X and Y.

(b) Compute $\mathbb{E}[X]$, $\mathbb{E}[Y]$ and $Var(X)$.

(c) Are the random variables X and Y independent? Justify.

(d) If Z and W are independent random variables, then $\mathbb{E}[ZW]=\mathbb{E}[Z]\mathbb{E}[W]$.
Is the opposite true? Prove or exhibit a counter example.

(e) Find the distribution function of X and represent its graph.

(f) Exhibit the distribution measure μ_X of X.

(g) Compute the distribution function of $X+Y$.

(h) Compute the distribution function of $X-Y$.

Exercise 4:

Suppose that X and Y are random variables with joint distribution given by:

$X \setminus Y$	1	0	−1
1	0	a	0
0	b	c	b
−1	0	a	0

where $a, b, c > 0$.

(a) Compute the marginal probability functions of X and Y.
Justify that $2a + 2b + c = 1$.

(b) Compute $\mathbb{E}[X]$, $\mathbb{E}[Y]$ and $Var(X)$.

(c) Verify that the random variable XY is such that $XY = 0$ almost surely.

(d) Are the random variables X and Y independent? Justify.

(e) If Z and W are independent random variables, then $\mathbb{E}[ZW] = \mathbb{E}[Z]\mathbb{E}[W]$.
Is the opposite true? Prove or exhibit a counter example.

(f) Take $c = 1/4$ and a, b such that $a = 2b$.

 (f_1) Find the distribution function of X and represent its graph.

 (f_2) Exhibit the distribution measure μ_X of X.

Exercise 5:

Let X be a random variable such that $X \sim \mathscr{U}[0,1]$. Compute the distribution of $Y = -\log(X)$.

Exercise 6:

Let X and Y be i.i.d. random variables with $X \sim \mathscr{U}[0,1]$. Compute the distribution of $Z = X/Y$.

Exercise 7:

Let X and Y have joint density given by $f(x, y)$. Show that

$$f_{X+Y}(u) = \int_{\mathbb{R}} f(u - t, t) dt.$$

Moreover, if X and Y are independent with densities f_X and f_Y, respectively, then

$$f_{X+Y}(u) = \int_{\mathbb{R}} f_X(t) f_Y(u - t) dt.$$

Exercise 8:

Let X be a r.v. with density $f(x) = \frac{1}{4} e^{-|x|/2}$, for $x \in \mathbb{R}$. Compute the distribution of $Y = |X|$.

Exercise 9:

Show that the function

$$F(x, y) = \begin{cases} 1 - e^{-(x+y)}, & x \geq 0 \quad \text{and} \quad y \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

is not the distribution function of a random vector.

Exercise 10:

Show that the function

$$F(x,y)=\begin{cases}(1-e^{-x})(1-e^{-y}), & x\geq 0 \quad \text{and} \quad y\geq 0\\ 0, & \text{otherwise}\end{cases}$$

is the distribution function of a random vector.

Exercise 11:

Let X and Y be i.i.d. random variables with uniform distribution on $[\theta-1/2,\theta+1/2]$, with $\theta\in\mathbb{R}$. Compute the distribution of $X-Y$.

Exercise 12:

Let $X_1,X_2,\ldots,X_n$ be i.i.d. random variables with Rayleigh distribution with parameter θ, that is, the density of X_1 is given by

$$f(x)=\begin{cases}\dfrac{x}{\theta^2}e^{-\dfrac{x^2}{2\theta^2}}, & x>0\\ 0, & \text{otherwise}\end{cases}$$

(a) Compute the joint density of $Y_1,\ldots,Y_n$, where for each $i=1,\ldots,n$ it holds that $Y_i=X_i^2$.

(b) Compute the distribution of $U=\min_{1\leq i\leq n}X_i$.

(c) Compute the distribution of $Z=X_1/X_2$.

Exercise 13:

Let $X_1,X_2,\ldots,X_n$ be independent random variables with exponential distribution with parameter $\alpha_1,\ldots,\alpha_n$, respectively.

(a) Compute the distribution of $Y=\min_{1\leq i\leq n}X_i$ and $Z=\max_{1\leq i\leq n}X_i$.

(b) Show that for each $p=1,\ldots,n$ it holds that

$$\mathbb{P}(X_p=\min_{1\leq i\leq n}X_i)=\frac{\alpha_p}{\alpha_1+\cdots+\alpha_n}.$$

(Hint: Consider the event $\{X_p<\min_{i\neq p}X_i\}$).

Exercise 14:

Let $X_1,X_2,\ldots,X_n$ be independent random variables with distribution functions $F_1,F_2,\cdots,F_n$ respectively. Find the distribution functions of the random variables $\min_{1\leq i\leq n}X_i$ and $\max_{1\leq i\leq n}X_i$.

Exercise 15:

Let X and Y be independent random variables each assuming the values 1 and -1 with probability $1/2$. Show that $\{X,Y,XY\}$ are pairwise independent but not totally independent.

Exercise list number 4 - Mathematical Expectation.

Exercise 1:

In each case, compute $\mathbb{E}(X)$ and $Var(X)$, if they exist:

(a) If X has probability function given on $k \in \{0,1\}$ by $\mathbb{P}(X=k) = p^k(1-p)^{1-k}$. That is X has Bernoulli distribution of parameter p.

(b) If X has probability function given on $k \in \{0,\cdots,n\}$ by $\mathbb{P}(X=k) = C_k^n p^k(1-p)^{n-k}$. That is X has Binomial distribution of parameter n and p.

(c) If X has probability function given on $k \in \{0,1,\cdots\}$ by $\mathbb{P}(X=k) = \frac{e^{-\alpha}\alpha^k}{k!}$, $\alpha > 0$. That is X has Poisson distribution of parameter α.

(d) If X has probability function given on $k \in \{0,1,\cdots\}$ by $\mathbb{P}(X=k) = p(1-p)^k$. That is X has Geometric distribution of parameter p.

(e) If X has probability density function given by $f(x) = \alpha e^{-\alpha x}\mathbf{1}_{[0,+\infty)}(x)$, with $\alpha > 0$. That is X has Exponential distribution with parameter α.

(f) If X has probability density function given by $f(x) = \frac{1}{b-a}\mathbf{1}_{[a,b]}(x)$ for $a, b \in \mathbb{R}$ with $a < b$. That is X has Uniform distribution in $[a,b]$.

(g) If X has probability density function given by $f(x) = \frac{1}{\pi(1+x^2)}$, $x \in \mathbb{R}$. That is X has Cauchy distribution.

(h) If X has probability density function given by $f(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$, $x \in \mathbb{R}$. That is X has Normal distribution.

Exercise 2:

Prove that:

(a) For any random variable X with distribution function F_X, it holds that

$$\mathbb{E}[X] = \int_0^{+\infty} 1 - F_X(x)dx - \int_{-\infty}^0 F_X(x)dx$$

(b) and for any $k \in \mathbb{N}$

$$\mathbb{E}[X^k] = k\int_0^{+\infty} (1-F_X(x))x^{k-1}dx - k\int_{-\infty}^0 F_X(x)x^{k-1}dx.$$

(c) If X is non-negative, then

$$\mathbb{E}[X] = \int_0^{+\infty} 1 - F_X(x)dx.$$

(d) If X is discrete and takes non-negative integer values, then

$$\mathbb{E}[X] = \sum_{n=1}^{+\infty} \mathbb{P}(X \geq n).$$

(e) If X has Exponential distribution with parameter $\lambda > 0$, then $\mathbb{E}[X^k] = k!/\lambda^k$, for any $k \in \mathbb{N}$.

(f) Let X and Y be random variables, such that Y is stochastically dominated by X, that is for all $x \in \mathbb{R}$ it holds that $F_X(x) \leq F_Y(x)$. Show that $\mathbb{E}[X] \geq \mathbb{E}[Y]$, if both expectations exist.

Exercise 3:

Show that:

(a) if X is a constant random variable, then $Var(X) = 0$.

(b) if $a \in \mathbb{R}$ then $Var(X+a) = Var(X)$.

(c) if $a, b \in \mathbb{R}$ then $Var(aX+b) = a^2 Var(X)$.

Exercise 4:

Prove:

(a) Basic Tchebychev's inequality:
If X is a non-negative random variable (that is $X \geq 0$), then for all $\lambda > 0$: $P(X \geq \lambda) \leq \frac{1}{\lambda}\mathbb{E}(X)$.

(b) Classical Tchebychev's inequality:
If X is an integrable random variable, then for all $\lambda > 0$: $\mathbb{P}(|X - \mathbb{E}(X)| \geq \lambda) \leq \frac{1}{\lambda^2} Var(X)$.

(b) Markov's inequality:
If X is a random variable, then for all $t > 0$ and $\lambda > 0$: $\mathbb{P}(|X| \geq \lambda) \leq \frac{1}{\lambda^t}\mathbb{E}(|X|^t)$.

Exercise 5:

(a) Let X be a non-negative random variable, that is $X \geq 0$, such that $\mathbb{E}(X) = 0$.
Show that $\mathbb{P}(X = 0) = 1$, that is, $X = 0$ almost surely.

(b) Let X be a random variable independent of itself.
Show that X is constant with probability 1 (that is, there exists a constant c such that $\mathbb{P}(X = c) = 1$).

Exercise 6:

Let $X_1, \cdots, X_n$ be integrable random variables, such that for $i \neq j$,

$$Cov(X_i, X_j) := \mathbb{E}[X_i X_j] - \mathbb{E}[X_i]\mathbb{E}[X_j] = 0.$$

Show that

$$Var(X_1 + \cdots + X_n) = Var(X_1) + \cdots + Var(X_n).$$

Exercise 7:

Let $X_1, \cdots, X_n$ be independent random variables with distribution function $F_{X_1}, \cdots, F_{X_n}$, respectively.

(a) Find the distribution function of $\max_{1 \leq j \leq n} X_j$ and $\min_{1 \leq j \leq n} X_j$.

(b) Suppose that the random variables are identically distributed with finite mean. Show that

$$\lim_{n \to +\infty} \frac{1}{n}\mathbb{E}\Big[\max_{1 \leq j \leq n} |X_j|\Big] = 0.$$

Exercise 8:

Let X and Y be random variables defined on a probability space $(\Omega, \mathscr{F}, P)$, both with finite expectation. Show that

(a) $\mathbb{E}[X+Y]=\mathbb{E}[X]+\mathbb{E}[Y]$.

(b) if X and Y are independent, then $\mathbb{E}[XY]=\mathbb{E}[X]\mathbb{E}[Y]$.

Exercise 9:

Let (X,Y) be a random vector with density function given by

$$f_{X,Y}(x,y)=\frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\exp\Big\{\frac{-1}{2(1-\rho^2)}\Big[\Big(\frac{x-\mu_1}{\sigma_1}\Big)^2-2\rho\Big(\frac{x-\mu_1}{\sigma_1}\Big)\Big(\frac{y-\mu_2}{\sigma_2}\Big)+\Big(\frac{y-\mu_2}{\sigma_2}\Big)^2\Big]\Big\}.$$

(a) Find the marginal distributions of X and Y.

(b) Assume that X and Y are independent. Compute the distribution of $X+Y$.

(c) Show that X and Y are independent if and only if $\rho=0$.

Exercise 10:

Let X and Y be random variables taking only the values 0 and 1. Show that, if $\mathbb{E}[XY]=\mathbb{E}[X]\mathbb{E}[Y]$ then X and Y are independent.

Exercise 11:

Let X and Y be random variables with finite variance. Show that, if $Var(X)\neq Var(Y)$ then $X+Y$ and $X-Y$ are not independent.

Exercise 12:

Let X and Y be i.i.d. random variables with Uniform distribution in $[0,1]$. Compute the expectation of $\min(X,Y)$ and $\max(X,Y)$.

Exercise 13:

Prove Wald's equation, that is, show that $\mathbb{E}[S_t]=E[\mathbb{N}_t]\mathbb{E}[X_1]$, where $S(t)$ is a compound stochastic process, or else, $S(t):=\sum_{i=1}^{N_t}X_i$, where N_t is a counting process (i.e. N_t takes values in $\mathbb{N}$) and $\{X_i\}_{i\geq 1}$ is a sequence of i.i.d. random variables and independent of N_t for all t.

Exercise 14:

Let X be a random variable and $F_X(\cdot)$ its distribution function. Prove that, for any $a\geq 0$, we have

$$\int_{\mathbb{R}}\Big(F_X(x+a)-F_X(x)\Big)dx=a.$$

Exercise 15:

Show that if $Cov(X,Y)=\sqrt{Var(X)}\sqrt{Var(Y)}$, then there exist constants a and b such that

$$\mathbb{P}(Y=aX+b)=1.$$

Exercise list number 5 - Convergence of sequences of random variables.

Exercise 1:

Let $(\mathscr{E}_n)_{n\geq 1}$ be random events on a probability space $(\Omega, \mathscr{F}, P)$. Show that

$$\mathbb{P}(\mathscr{E}_n) \xrightarrow[n\to+\infty]{} 0 \Leftrightarrow \mathbf{1}_{\mathscr{E}_n} \xrightarrow[n\to+\infty]{} 0, \quad \text{in probability.}$$

Exercise 2:

Let $(X_n)_{n\geq 1}$ be a sequence of random variables.
Show that if $\mathbb{E}(X_n) \xrightarrow[n\to+\infty]{} \alpha$ and $Var(X_n) \xrightarrow[n\to+\infty]{} 0$, then $X_n \xrightarrow[n\to+\infty]{} \alpha$, in probability.

Exercise 3:

(a) Let $(X_n)_{n\geq 1}$ be a sequence of random variables such that for each $n \geq 1$ it holds that

$$\mathbb{P}(X_n = 1) = 1/n \quad \text{and} \quad \mathbb{P}(X_n = 0) = 1 - 1/n.$$

Show that

$$X_n \xrightarrow[n\to+\infty]{} 0, \quad \text{in probability.}$$

(b) Now suppose that for each $n \geq 1$ we have that $\mathbb{P}(X_n = 1) = p_n$ and $\mathbb{P}(X_n = 0) = 1 - p_n$, and suppose that $(X_n)_{n\geq 1}$ are independent. Show that:

$$\begin{aligned}
&(1) \quad X_n \xrightarrow[n\to+\infty]{} 0, \quad \text{in probability} \Leftrightarrow p_n \xrightarrow[n\to+\infty]{} 0.\\
&(2) \quad X_n \xrightarrow[n\to+\infty]{} 0, \quad \text{in } \mathbb{L}^p \Leftrightarrow p_n \xrightarrow[n\to+\infty]{} 0.\\
&(3) \quad X_n \xrightarrow[n\to+\infty]{} 0, \quad \text{almost everywhere} \Leftrightarrow \sum_{n\geq 1} p_n < +\infty.
\end{aligned}$$

(c) Justify if in (a) the sequence $(X_n)_{n\geq 1}$ converges almost everywhere to 0.

Exercise 4:

Prove the Tchebychev's weak law:

Let $(X_n)_{n\geq 1}$ be a sequence of random variables pairwise independent, with finite variance and uniformly bounded, i.e. there exists a constant $c < +\infty$ such that $Var(X_n) \leq c$ for all $n \geq 1$. Then,

$$\frac{S_n - \mathbb{E}(S_n)}{n} \to_{n\to+\infty} 0, \quad \text{in probability,}$$

where $S_n = \sum_{j=1}^n X_j$ is the sequence of the partial sums of $(X_n)_{n\geq 1}$.

Exercise 5:

Prove the Bernoulli's Law of Large Numbers:
Consider a sequence of independent Binomial experiments, with the same probability p of success in each experiment. Let S_n be the number of successes in the first n experiments. Then,

$$\frac{S_n}{n} \to_{n\to+\infty} p, \quad \text{in probability.}$$

Exercise 6:

Consider a sequence of independent Binomial experiments with probability p_n of success in the n-th trial. For $n \geq 1$, let $X_n = 1$ if the n-trial is a success, and $X_n = 0$ otherwise. Show that

(a) If $\sum_{n\geq 1} p_n = +\infty$, then $\mathbb{P}(\sum_{n\geq 1} X_n = +\infty) = 1$, (there are an infinite number of successes a.e.).

(b) If $\sum_{n\geq 1} p_n < +\infty$, then $\mathbb{P}(\sum_{n\geq 1} X_n < \infty) = 1$, (there are a finite number of successes a.e.).

Exercise 7:

Let $(X_n)_{n\geq 1}$ be a sequence of independent random variables such that for each $n \geq 1$ it holds that

$$\mathbb{P}(X_n = e^n) = \frac{1}{n+1} \quad \text{and} \quad \mathbb{P}(X_n = 0) = 1 - \frac{1}{n+1}.$$

Analyze the convergence of $(X_n)_{n\geq 1}$ to $X = 0$ in the case of

(a) convergence in probability.

(b) convergence in $\mathbb{L}^p$, for $p > 0$.

(c) convergence almost everywhere.

(d) convergence in distribution.

Exercise 8:

Let $(X_n)_{n\geq 1}$ be a sequence of independent random variables such that for each $n \geq 1$ it holds that

$$\mathbb{P}(X_n = 1) = \frac{1}{2^n} \quad \text{and} \quad \mathbb{P}(X_n = 0) = 1 - \frac{1}{2^n}.$$

Show that $X_n \xrightarrow[n\to+\infty]{} 0$,

(a) in probability.

(b) in $\mathbb{L}^p$, for $p > 0$.

(c) almost everywhere.

(d) in distribution.

Exercise 9:

Let X and Y be random variables defined on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. The covariance between X and Y is defined by

$$Cov(X,Y) := \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y].$$

Let $X_1, \cdots, X_n$ be uncorrelated random variables, i.e. such that $Cov(X_i, X_j) = 0$, for $i \neq j$, with $\mathbb{E}[X_i] = \mu$ and $Var(X_i) \leq C < +\infty$, for all $i \geq 1$, where C is a constant. If $S_n := X_1 + \cdots + X_n$, show that

(a) $\mathbb{E}[S_n] = n\mu$ and $Cov(X,Y) = \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])]$.

(b) $Var(S_n) = Var(X_1) + \cdots + Var(X_n)$.

(c) $\frac{S_n}{n} \xrightarrow[n\to+\infty]{} \mu$, in $\mathbb{L}^2$ and in probability.

Exercise 10:

Let $(X_n)_{n\geq 2}$be a sequence of independent and identically distributed random variables such that X_1 has exponential distribution with parameter 1. For each $n \geq 2$ let $Y_n = X_n/\log(n)$. Analyze the convergence of $(Y_n)_{n\geq 2}$ to $Y = 0$ in the case of

(a) convergence in probability.

(b) convergence in $\mathbb{L}^1$.

(c) convergence almost everywhere.

(d) convergence in distribution.

Exercise 11:

Let $X_1, X_2, X_3 \ldots$ be independent random variables with $X_n \sim \mathcal{U}[0, a_n]$, with $a_n > 0$. Show that

(a) If $a_n = n^2$, then, with probability 1, only a finite number of X_n takes values less than 1.

(b) If $a_n = n$, then, with probability 1, an infinite number of X_n takes values less than 1.

Exercise 12:

Let $(X_n)_{n\geq 1}$ be a sequence of i.i.d. random variables such that $X_1 \sim \mathcal{U}[0,1]$. Show that n^{-X_n} converges to 0 in probability but it does not converge to 0 almost surely.

Exercise 13:

Let $(X_n)_{n\geq 1}$ be a sequence of random variables such that for $n \in \mathbb{N}$ it holds that

$$\mathbb{P}(X_n = n^2) = 1/n^2 \quad \text{and} \quad \mathbb{P}(X_n = 0) = 1 - 1/n^2.$$

Show that X_n converges almost surely (find the limit X) but $\mathbb{E}[X_n^m]$ does not converge to $\mathbb{E}[X^m]$, for all $m \in \mathbb{N}$.

Exercise 14:

Let $(X_n)_{n\geq 1}$ be a sequence of i.i.d. random variables such that $X_1 \sim \mathcal{U}[0,1]$. Find the limit in probability of $\left(\prod_{k=1}^n X_k\right)^{1/n}$.

Exercise 15:

Let $(X_n)_{n\geq 1}$ be a sequence of i.i.d. random variables such that $\mathbb{E}[X_1] = 1$ and $Var(X_1) = 1$. Show that

$$\frac{\sum_{k=1}^n X_k}{\sqrt{n \sum_{k=1}^n X_k^2}} \to_{n\to+\infty} 1/\sqrt{2}$$

in probability.

Exercise 16:

Let $(X_n)_{n\geq 1}$ be a sequence of independent random variables such that $\mathbb{E}[X_n] = 0$ and $\mathbb{E}[X_n^2] = 1$ for all $n \in \mathbb{N}$. Let $S_n := X_1 + \cdots + X_n$ and for all $x \in \mathbb{R}$ let $\varphi(x) = \int_{-\infty}^x \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy$. If $\mathbb{P}(S_n \leq \sqrt{n}x) \to \varphi(x)$ for all $x \in \mathbb{R}$, show that $\limsup_{n\to+\infty} \frac{S_n}{\sqrt{n}} = +\infty$ almost everywhere.

Exercise 17:

Show that if X_n converges to X in probability, as $n \to +\infty$, and if $g : \mathbb{R} \to \mathbb{R}$ is a continuous function, then $g(X_n)$ converges to $g(X)$ in probability, as $n \to +\infty$.

Exercise 18:

Let $(X_n)_{n\geq 1}$ be a sequence of independent random variables with distribution function F_n. Prove that, $\mathbb{P}(\lim_n X_n = 0) = 1$ if and only if $\forall \varepsilon > 0$,

$$\sum_{n\geq 1}\{1 - F_n(\varepsilon) + F_n(-\varepsilon^-)\} < +\infty.$$

Exercise 19:

If $\sum_{n\geq 1} \mathbb{P}(|X_n| > n) < \infty$, then $\limsup_n \frac{|X_n|}{n} \leq 1$ almost everywhere.

Exercise 20:

(a) Let X and Y be independent random variables with laws $X \sim$Poisson(λ_1) and $Y \sim$Poisson(λ_2). What is the law of $X + Y$?

(b) Let Z be a random variable with law Poisson(λ), and let $\xi_1, \xi_2, \ldots$ be i.i.d. Bernoulli(p) random variables, independent of Z. Define $X := \sum_{j=1}^{Z} \xi_i$. Show that X has law Poisson$(p\lambda)$.

Remark: Item (b) is know as the *Poisson coloring theorem*. You can think you have a Poisson number of balls, and color each ball either red (with probability p) or blue (with probability $1-p$), independently. Then the number of red balls is also Poisson distributed. This is one of the basic results in the theory of *Poisson Point Process*.

Exercise 21:

(a) Let X be a random variable with law Exp(λ), and let $t, s > 0$. Prove that

$$\mathbb{P}(X > t + s | X > s) = P(X > t).$$

This property is called "lack of memory of the exponential distribution".

(b) Let Y_n be a geometric random variable with success probability $\frac{\lambda}{n}$ (assume n large enough, so that $\frac{\lambda}{n} < 1$). Show that $\frac{Y_n}{n}$ converges weakly to an Exp(λ) distribution.

Exercise list number 6 - Characteristic functions.

Exercise 1:

Compute the characteristic function of each one of the following random variables:

(a) X such that $\mathbb{P}(X=a)=1$ and $\mathbb{P}(X\neq a)=0$.

(b) X such that $\mathbb{P}(X=1)=1/2$ and $\mathbb{P}(X=-1)=1/2$.

(c) X with Bernoulli distribution with parameter p.

(d) X with Binomial distribution with parameter n and p.

(e) X with Geometric distribution with parameter p.

(f) X with Poisson distribution with parameter λ.

(g) X with exponential distribution with parameter λ.

(h) X with uniform distribution on $[-a,a]$, with $a>0$.

(i) X with triangular distribution on $[-a,a]$, with $a>0$.

(j) X with Gaussian distribution with mean μ and variance σ^2.

Exercise 2:

(a) Show that for X and Y independent random variables it holds that $\varphi_{X+Y}=\varphi_X\varphi_Y$.

(b) Show that if φ is a characteristic function, then $|\varphi|^2$ is also a characteristic function.

Exercise 3:

Let φ be a characteristic function. Show that $\psi(t)=e^{\lambda(\varphi(t)-1)}$ with $\lambda>0$ is also a characteristic function.

Suggestion: Let $N, X_1, X_2, \cdots$ be independent random variables with $N\sim$Poisson(λ) and $(X_n)_{n\geq 1}$ identically distributed with $\varphi_{X_n}=\varphi$ for all $n\geq 1$. Let $Y:=S_N$, with $S_n=X_1+\cdots+X_n$. Then $\varphi_Y=\psi$.

Exercise 4:

Let φ_X be a characteristic function of a random variable X with Binomial distribution with parameter n and p. Find φ_X and $\mathbb{E}[X]$ and verify that $i^{-1}\varphi_X'(0)=\mathbb{E}[X]=np$.

Exercise 5:

Let $(X_n)_{n\geq 1}$ be a sequence of random variables with Uniform distribution $\mathcal{U}[-n,n]$. Find φ such that

$$\varphi_n(t)\xrightarrow[n\to+\infty]{}\varphi(t),$$

for all $t\in\mathbb{R}$ where for each $n\geq 1$, φ_n is the characteristic function of X_n. Verify if φ is a characteristic function.

Exercise 6:

(a) Show that if $Y:=aX+b$ for $a,b\in\mathbb{R}$ and $a\neq 0$ then $\varphi_Y(t):=e^{itb}\varphi_X(at)$.

(b) Is $\varphi(t):=\mathbf{1}_{[0,\infty)}(t)$ a characteristic function? Justify.

(c) Is $\varphi(t):=t\mathbf{1}_{[0,1]}(t)+\mathbf{1}_{[1,\infty)}(t)$ a characteristic function? Justify.

(d) Show that X is a symmetric if and only if its characteristic function φ_X, takes values in $\mathbb{R}$.

(e) Let $\varphi(t)=\frac{1+\cos(3t)}{2}$. Find X such that φ is its characteristic function.

Exercise 7:

(a) Using characteristic functions show that if X and Y are independent and identically distributed random variables and if $X \sim \mathcal{N}(0,1)$ then $X+Y \sim \mathcal{N}(0,2)$.

(b) Obtain the previous result using convolutions. Justify.

(c) Compute the 3-rd centered moment of the random variable $X+Y$, i.e. compute $\mathbb{E}[(X+Y)^3]$. Suggestion: use characteristic functions.

(d) Let $X_1, \cdots, X_n$ be independent and identically distributed random variables such that $X_1 \sim \mathcal{N}(0,1)$. Using characteristic functions, show that

$$\frac{S_n}{n} \xrightarrow[n\to+\infty]{} 0,$$

in **probability**, where $S_n := X_1 + \cdots + X_n$.

Exercise 8:

Let $X_1, \cdots, X_n$ be independent random variables with Poisson distribution with parameter $\lambda_1, \cdots, \lambda_n$, respectively, where $\lambda_i > 0$, for all $i \geq 1$.

(a) Verify that $\mathbb{E}[X_1] = \lambda_1$.

(b) Compute the characteristic function φ_{X_1} of X_1.

(c) Verify that $d_t \log(\varphi_{X_1}(t)) = \lambda_1 i e^{it}$ and conclude that $i^{-1}\varphi'_{X_1}(0) = \mathbb{E}[X_1]$.

(d) Compute the characteristic function of $S_n = X_1 + \cdots + X_n$.

Exercise 9:

(a) Let X be a constant random variable and let φ_X be its characteristic function. Show that $|\varphi_X(t)|^2 = 1$ for all $t \in \mathbb{R}$.

(b) Let X be a random variable independent of itself. Show that X is constant a.e.

(c) Let X be a symmetric random variable that takes only two values θ and $-\theta$, with $\theta > 0$. Show that there is no $\theta \in \mathbb{R}$ such that $\varphi_X(t) = 1$ for all $t \in \mathbb{R}$ where φ_X denotes the characteristic function of X. Show that $\varphi''_X(0) = -\theta^2$. Conclude that $Var(X) = \theta^2$.

Exercise 10:

Find the distribution of the random variable $X+Y+Z$, knowing that X, Y and Z are independent and identically distributed random variables and such that X has Bernoulli distribution with parameter p, i.e. X induces the measure $\mu_X := p\delta_{\{1\}} + (1-p)\delta_{\{0\}}$.
Solve the exercise in two different ways: using the convolution and characteristic functions.

Exercise 11:

(a) Let X be a symmetric random variable that takes the values $a \neq b \neq c$. Knowing that $\mathbb{P}(X=0) = 1/5$, compute φ_X i.e. the characteristic function of X.

(b) Verify that there is no $a \in \mathbb{R}$ such that $\varphi_X(t) = 1$ for all $t \in \mathbb{R}$.

(c) Compute $\varphi'_X(t)$ and verify that $i^{-1}\varphi'_X(0) = \mathbb{E}[X]$.

(d) Find a such that $\varphi''_X(0) = -1$. Conclude that $Var(X) = 1$.

Exercise 12:

Justify if $\varphi(t) := \frac{e^{ita}+1}{2}$ is the characteristic functions of a symmetric random variable?
Find the random variable whose characteristic function is φ.

Exercise 13:

Find the distribution of the random variable $X+Y$, knowing that X has Poisson distribution of parameter λ_1 and Y is independent of X and has Poisson distribution of parameter λ_2. Solve in two different ways: using the convolution and characteristic functions.

Exercise 14:

Let X and Y be independent and identically distributed random variables such that X induces the measure $\mu_X := p\delta_{\{1\}} + q\delta_{\{-1\}}$ where $p+q=1$.

(a) Compute the characteristic function of X.

(b) Show that X is symmetric if and only if $p=1/2$.

(c) Take $p=1/2$. Let φ_{X+Y} be the characteristic function of the random variable $X+Y$. Verify that $\varphi_{X+Y}(t) := cos^2(t)$, for all $t\in\mathbb{R}$.

(d) Using the convolution, determine the distribution function of the random variable $X+Y$. Show that $X+Y$ is symmetric if and only if $p=1/2$. In this case, compute again the characteristic function of the random variable $X+Y$ and conclude that for all $t\in\mathbb{R}$

$$cos^2(t) := \frac{1+cos(2t)}{2}.$$

Exercise 15:

Let X and Y be independent and identically distributed random variables with $X\sim\mathcal{N}(0,1)$.

(a) Using characteristic functions and the convolution, show that $X+Y\sim\mathcal{N}(0,2)$.

(b) Show, using characteristic functions, that if $Z := \sigma X+\mu$ then $Z\sim\mathcal{N}(\mu,\sigma^2)$.

(c) Let φ_Z be the characteristic function of Z. Compute $|\varphi_Z|^2$ and verify that $|\varphi_Z|^2\leq 1$. Is the random variable Z symmetric?

(d) Show that $i^{-1}\varphi_Z'(0) := \mu$ and that $-\varphi_Z''(0) = \mu^2+\sigma^2$. Conclude that $Var(Z)=\sigma^2$.

Exercise 16:

(a) Let X be a random variable with exponential distribution with parameter $a>0$. Compute $\varphi_X'(t)$, where φ_X is the characteristic function of X and verify that $i^{-1}\varphi_X'(0)=\mathbb{E}[X]$.

(b) Find a such that $\varphi_X''(0) = -1/8$. Compute $Var(X)$.

Exercise 17:

(a) Find the random variable X such that $\varphi(t) := \cos(t)$ is its characteristic function. Justify.

(b) Show that a symmetric random variable has all its odd moments equal to zero.

(c) Is $\varphi(t) := \mathbf{1}_{[-1,1]}(t)$ a characteristic function?

(d) Justify if $\varphi(t) := \frac{e^{it}+1}{2}$ is the characteristic function of a symmetric random variable? Find the random variable whose characteristic function is φ. Compute $|\varphi|^2$.

Exercise 18:

Using characteristic functions, show that for $g:\mathbb{R}\to\mathbb{R}$ a continuous function, if

$$X_n \xrightarrow[n\to+\infty]{} X, \quad \text{weakly}$$

then

$$g(X_n) \xrightarrow[n\to+\infty]{} g(X), \quad \text{weakly.}$$

Exercise 19:

Using characteristic functions prove Slutsky's Theorem:

Let $(X_n)_{n\geq 1}$ and $(Y_n)_{n\geq 1}$ be two sequences of random variables and let X be a random variable. Suppose that

$$X_n \xrightarrow[n\to+\infty]{} X, \quad \text{weakly} \quad and \quad Y_n \xrightarrow[n\to+\infty]{} c, \quad \text{in} \quad \text{probability},$$

where c is a constant. Then

(a)

$$X_n + Y_n \xrightarrow[n\to+\infty]{} X + c, \quad \text{weakly}.$$

(b)

$$X_n - Y_n \xrightarrow[n\to+\infty]{} X - c, \quad \text{weakly}.$$

(c)

$$X_n Y_n \xrightarrow[n\to+\infty]{} Xc, \quad \text{weakly}.$$

(d) if $c \neq 0$ and $\mathbb{P}(Y_n \neq 0) = 1$, for all $n \geq 1$, then $\dfrac{X_n}{Y_n} \xrightarrow[n\to+\infty]{} \dfrac{X}{c}$, weakly.

Exercise 20:

Show, using characteristic functions that if $(X_n)_{n\geq 1}$ is a sequence of i.i.d.r.v. with $\mathbb{E}(X_1) = \mu < \infty$, then $\frac{S_n}{n} \xrightarrow[n\to+\infty]{} \mu$, in probability, where $S_n = \sum_{j=1}^n X_j$.

Exercise 21:

(a) Show, using characteristic functions that if $X \sim B(m,p)$ and $Y \sim B(n,p)$, and if X and Y are independent then $X + Y \sim B(n+m,p)$.

(b) Show that if X has standard Cauchy distribution, then $\varphi_{2X} = (\varphi_X)^2$. Use (without showing) that

$$\frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{\cos(tx)}{1+x^2}\, dx = e^{-|t|}.$$

(c) It is true that if X and Y are independent random variables then $\varphi_{X+Y} = \varphi_X \varphi_Y$. And the reciprocal, is it true? Prove and present a counter-example.

Exercise 22:

(a) Let $\varphi(t) = \cos(at)$ with $a > 0$. Show that φ is a characteristic function.

(b) Let $\varphi(t) = \cos^2(t)$. Find X such that φ is its characteristic function.

Exercise 23:

Let X and Y be i.i.d.r.v. with $\mathbb{E}(X) = 0$ and $Var(X) = 1$. Show that if $X + Y$ and $X - Y$ are independent then $X, Y \sim \mathcal{N}(0,1)$.

Exercise list number 7 - Martingales.

Exercise 1: Show that:

(a) if $(X_n)_{n\geq 1}$ is a sequence of independent r.v. with $\mathbb{E}[X_n]=0$ for all $n\geq 1$, then $(S_n,\mathscr{F}_n)_{n\geq 1}$ where $S_n=\sum_{j=1}^n X_j$ and $\mathscr{F}_n=\sigma(X_1,\cdots,X_n)$ is a martingale

(b) if $(X_n)_{n\geq 1}$ is a sequence of independent r.v. with $\mathbb{E}[X_n]=1$ for all $n\geq 1$, then $(\tilde{X}_n,\mathscr{F}_n)_{n\geq 1}$ where $\tilde{X}_n=\prod_{j=1}^n X_j$ and $\mathscr{F}_n=\sigma(X_1,\cdots,X_n)$, is a martingale.

(c) given an integrable r.v. X, that is with $\mathbb{E}[|X_n|]<+\infty$ and a set of σ-algebras $\mathscr{F}_0\subsetneqq\mathscr{F}_1\subsetneqq\cdots\subsetneqq\mathscr{F}_n$, then $(X_n,\mathscr{F}_n)_{n\geq 1}$ where $X_n=\mathbb{E}[X|\mathscr{F}_n]$ is a martingale.

Exercise 2: Show that:

(a) if $(X_n)_{n\geq 1}$ is a sequence of non-negative integrable r.v., then $(S_n,\mathscr{F}_n)_{n\geq 1}$ where $S_n=\sum_{j=1}^n X_j$ and $\mathscr{F}_n=\sigma(X_1,\cdots,X_n)$ is a submartingale.

(b) if $(X_n,\mathscr{F}_n)_{n\geq 1}$ i a martingale and $g:\mathbb{R}\to\mathbb{R}$ is a convex function with $\mathbb{E}[|g(X_n)|]<+\infty$ for all $n\geq 1$, then $(g(X_n),\mathscr{F}_n)_{n\geq 1}$ is a submartingale.

Exercise 3: Let $(X_n)_{n\geq 1}$ be i.i.d. r.v. with $\mathbb{P}(X_1=1)=p$ and $\mathbb{P}(X_1=-1)=q$ with $p+q=1$. If $p\neq q$, show that if $S_n=\sum_{j=1}^n X_j$ and $\mathscr{F}_n=\sigma(X_1,\cdots,X_n)$, then

(a) $(Y_n,\mathscr{F}_n)_{n\geq 1}$ is a martingale, where $Y_n=\left(\frac{q}{p}\right)^{S_n}$.

(b) $(Z_n,\mathscr{F}_n)_{n\geq 1}$ is a martingale, where $Z_n=S_n-n(p-q)$.

Exercise 4: Show that if $(X_n)_{n\geq 1}$ is a sequence of i.i.d. r.v. with $\mathbb{E}[X_n]=0$ and $Var(X_n)=\sigma^2$ for all $n\geq 1$, then $(\mathscr{W}_n,\mathscr{F}_n)_{n\geq 1}$ is a martingale, where $\mathscr{F}_n=\sigma(X_1,\cdots,X_n)$ and

(a)

$$\mathscr{W}_n=\Big(\sum_{j=1}^n X_j\Big)^2-n\sigma^2.$$

(b)

$$\mathscr{W}_n=\frac{e^{\lambda\sum_{j=1}^n X_j}}{(E[e^{\lambda X_1}])^n}.$$

Exercise 5: Let $(X_n)_{n\geq 1}$ be a sequence of i.i.d. r.v. that take values on a finite set $\mathscr{I}$. For each $y\in\mathscr{I}$, let $f_0(y)=\mathbb{P}(X_1=y)$ and let $f_1:\mathscr{I}\to[0,1]$ be a non-negative function such that $\sum_{y\in\mathscr{I}}f_1(y)=1$. Show that $(\mathscr{W}_n,\mathscr{F}_n)_{n\geq 1}$ is a martingale, where $\mathscr{F}_n=\sigma(X_1,\cdots,X_n)$ and

$$\mathscr{W}_n=\frac{f_1(X_1)\cdots f_1(X_n)}{f_0(X_1)\cdots f_0(X_n)}.$$

The r.v. $\mathscr{W}_n$ are known as likelihood ratios.

Exercise 6: Let $(X_n, \mathscr{F}_n)_{n\geq 1}$ be a martingale.

(a) Show that, for all $n < m$ it holds that $X_n = E[X_m|\mathscr{F}_n]$.

(b) Conclude that $\mathbb{E}[X_1] = \mathbb{E}[X_n]$ for all $n \geq 1$.

(c) For each $n \geq 2$ let $Y_n = X_n - X_{n-1}$ and take $Y_1 = X_1$. We observe that Y_n is called the increment of the martingale. Show that $\mathbb{E}[Y_n] = 0$ for all $n \geq 0$.

(d) Assume that $\mathbb{E}[X_n^2] < +\infty$ for all $n \geq 1$. Show that the increments of the martingale are non correlated.

(e) Show that $Var(X_n) = \sum_{j=1}^n Var(Y_j)$.

Exercise 7: Let $(X_n, \mathscr{F}_n)_{n\geq 1}$ and $(Y_n, \mathscr{F}_n)_{n\geq 1}$ be two martingales with $X_1 = Y_1 = 0$. Show that

$$\mathbb{E}[X_n Y_n] = \sum_{k=2}^{n} \mathbb{E}[(X_k - X_{k-1})(Y_k - Y_{k-1})].$$

Exercise 8: Let $(X_n, \mathscr{F}_n)_{n\geq 1}$ be a martingale (or submartingale) and τ a Markov time (with respect to $\mathscr{F}_n$). Then, the stopping time

$$X^\tau = (X_{\min\{n,\tau\}}, \mathscr{F}_n)$$

is also a martingale (or a submartingale).

Exercise 9:

(a) Prove Wald's equality. Let $(X_n)_{n\geq 1}$ be a sequence of integrable i.i.d. r.v. and let τ be a stopping time with respect to $\mathscr{F}_n = \sigma(X_1, \cdots, X_n)$ and $\mathbb{E}[\tau] < \infty$. Then, $\mathbb{E}[X_1 + \cdots + X_\tau] = \mathbb{E}[X_1]\mathbb{E}[\tau]$.

(b) Analyze the case in which $\mathbb{P}(X_1 = 1) = 1/2 = \mathbb{P}(X_1 = -1)$ and $\tau = \inf\{n \geq 1 : X_1 + \cdots + X_\tau = 1\}$. What do you conclude about $\mathbb{E}[\tau]$?

Exercise 10: Let $(X_n)_{n\geq 1}$ be a sequence of i.i.d.r.v. such that $\mathbb{P}(X_1 = 1) = p = 1 - \mathbb{P}(X_1 = -1)$. Interpret $X_n = 1$ as a success and $X_n = -1$ as the lost of a player in its n-th play. Assume that the player can win or lose in the n-th play the amount V_n (so that V_n is the amount of the bet in the n-th play). The total amount of the player at the n-th play is given by $Y_n = \sum_{i=1}^n X_i V_i$. Assume that V_i is predictable with respect to $\mathscr{F}_n = \sigma(X_1, \cdots, X_n)$.

a) Verify in which conditions the game is fair, favorable or unfavorable. In each case, verify if $(Y_n, \mathscr{F}_n)_n$ is a martingale, sub-martingale or supermartingale.

b) Now consider the following strategy $V_1 = 1$ and $V_n = 2^{n-1}\mathbf{1}_{\{X_1=-1,\cdots,X_{n-1}=-1\}}$. Say by words what means that strategy. Is $(V_n)_n$ predictable with respect to $\mathscr{F}_n$? Let $\tau = \inf\{n \geq 1 : Y_n = 1\}$. Take $p = 1/2$, compute the probability function of τ and express $\mathbb{P}(\tau < \infty)$. Compute $\mathbb{E}[Y_\tau]$. What can you say about the game?