Martingales in Ruin theory

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Introduction

Introduction

- Useful tools
- Cramer Lundberg model
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- Observations

Useful tools

Chebychev's inequality:

X r.v with finite variance, $\lambda > 0$ then:

$$\mathbb{P}[\mid X - \mathbb{E}\, X \mid \geq \lambda] \leq \frac{VarX}{\lambda^2}$$

Poisson Process

(Karr, 1993, p. 91; Kulkarni, 1995, p. 203)
A counting process $\{N_t\}_{t \geq 0}$ is said to be a (homogenous) *Poisson Process* with *rate* λ if:

- $\{N_t\}_{t \geq 0}$ has independent and stationary increments
- $N_t \sim \mathit{Poisson}(\lambda t)$

Wald's identities

For the mean: N_t r.v. assuming positive integer values, X_i sequence of i.i.d. r.v.'s, $X_i \perp\!\!\!\perp N$, then:

$$\mathbb{E} \sum_{i=1}^{N_t} X_i = \mathbb{E}\, N_t \, \mathbb{E}\, X$$

For the variance:

$$Var \sum_{i=1}^{N_t} X_i = \mathbb{E}\, N_t \, VarX + \mathbb{E}^2\, XVarN_t$$

Doob's Optional stopping theorem

Let $(\mathscr{F}_t)_{t\geq 0}$ be a filtration defined on the probability space $(\Omega, \mathscr{F}, \mathbb{P})$, and let $(M_t)_{t\geq 0}$ be a stochastic process adapted to the filtration $(\mathscr{F}_t)_{t\geq 0}$ whose paths are right continuous and locally bounded. The following properties are equivalent:

- $(M_t)_{t\geq 0}$ is a martingale w.r.t. $(\mathscr{F}_t)_{t\geq 0}$
- For any almost surely bounded stopping time T of the filtration $(\mathscr{F}_t)_{t\geq 0}$ such that $\mathbb{E} \mid M_T \mid < \infty$ we have $\mathbb{E}\, M_T = \mathbb{E}\, M_0$

Cramer Lundberg model

The evolution of the capital $U = (U_t)_{t \geq 0}$ of a certain insurance company takes place in a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ as follows:
The initial capital is $U_0 = u > 0$. Insurance *payments* arrive continuously at a *constant* rate $c > 0$ and claims are received at random times $0 < T_1 < T_2 < \dots$, where the amounts to paid out at these times are described by nonnegative r.v.'s $X_1, X_2, \dots$

Cramer Lundberg model

i.e., the capital U_t at time $t > 0$ is determined by the formula

$$U_t = u + ct - S_t$$

where

- $S_t = \sum_{i=1}^{N_t} X_i$ represents the total ammount of claims
- N_T is the number of claims up to time t

We will assume that

- $(X_i)_{i \geq 1}$ is an *i.i.d* sequence of r.v.'s
- $(X_i)_{i \geq 1} \perp\!\!\!\perp N_t \quad \forall t \geq 0$
- $(N_t)_{t \geq 0}$ is a (Poisson) counting process
- $\mathbb{E}\, X > 0$

Cramer Lundberg model

One of the main questions relating to the operation of an insurance company is the calculation of the *probability of ruin*, $\mathbb{P}(T < \infty)$, and the probability of ruin before time t, $P(T \leq t)$.

By *Wald's identity*, notice that

$$\begin{aligned} \mathbb{E}(U_t - U_0) &= ct - \mathbb{E}\, S_t = ct - \mathbb{E} \sum_{i=1}^{N_t} X_i \\ &= ct - \mathbb{E}\, N_t \, \mathbb{E}\, X \\ &= t(c - \lambda \, \mathbb{E}\, X) \end{aligned}$$

Cramer Lundberg model

Thus, in the case under consideration, a natural requirement for an insurance company to operate with a clear profit is that $\mathbb{E}\, U_t - U_0 > 0$, i.e., $c > \lambda \mathbb{E}\, X$

We also define the *time of ruin*, T:

$$T := \inf\{t \geq 0 : U_t \leq 0\}$$

i.e., the first time at which the insurance company's capital becomes less than or equal to zero. Of course, $U_t \geq 0 \quad \forall t \geq 0 \Rightarrow T = \infty$

Cramer Lundberg model

Our main objective is to derive an upper bound for the *probability of ruin*, which we shall do with two different approaches: a more intuitive and longer approach, and a martingale approach.

First Approach

First Approach

Given that we have an expression for the (poisson) point process N_t, seems reasonable to try and compute the moment generating function of our process U_t - m_{U_t}- , which will be a function of the m.g.f of N_t, which, since $\{X_i\}_i$ are *i.i.d* will surely be a function of m_X. Conditioning on *the time of ruin* we can get an expression depending explicitly on m_X and $\psi(u)$, which might proove to be useful.

First Approach

The trick is to decompose $\mathbb{E}\, e^{-rU_t}$ in order to get $\mathbb{P}(T \leq t)$:

$$\mathbb{E}\, e^{-rU_t} = \mathbb{E}[e^{-rU_t} \mid T \leq t]\, \mathbb{P}(T \leq t) + \mathbb{E}[e^{-rU_t} \mid T > t]\, \mathbb{P}(T > t) \quad (1)$$

First Approach

But also simplify $\mathbb{E}[e^{U_t}]$ in order to get $m_X(r)$:

$$\mathbb{E}[e^{-rU_t}] = e^{-ru-crt}\,\mathbb{E}\,e^{rS_t} = e^{-(ru+crt)}m_{S_t}(r)$$

$$m_{S_t}(r) = \mathbb{E}\,e^{rS_t} = \mathbb{E}\,e^{r\sum_{i=0}^{N_t}} = \sum_n \mathbb{E}[e^{r\sum_{i=0}^{N_t}} \mid N_t = n]\,\mathbb{P}(N_t = n)$$

$$= \sum_n \mathbb{E}[e^{r\sum_{i=0}^{N_t}} \mid N_t = n]\frac{(\lambda t)^n}{n!}e^{-\lambda t}$$

$$= \sum_n m_X^n(r)\frac{(\lambda t)^n}{n!} = e^{\lambda t(m_X(r)-1)}$$

$$\Rightarrow \mathbb{E}[e^{-rU_t}] = e^{-(ru+crt)}e^{\lambda t(m_X(r)-1)}$$

$$= \exp[-ru - crt + \lambda t(m_X(r) - 1)]$$

First Approach

Our life would be much easier if $t = 0$ or if there was $r : g(r) = -crt + \lambda t(m_X(r) - 1) = 0$. In fact, there is an unique solution, which we shall denote by R.
Therefore, for such R we get:

$$\mathbb{E}[e^{-RU_t}] = e^{-Ru}$$

which is NOT a function of time. Therefore we can take the limit and get:

$$e^{-Ru} = \lim_{t\to\infty} \mathbb{E}[e^{-rU_t} \mid T \leq t]\,\mathbb{P}(T \leq t) + \lim_{t\to\infty} \mathbb{E}[e^{-rU_t} \mid T > t]\,\mathbb{P}(T > t)$$

First Approach

- $U_t - U_T = c(t-T) - (S_t - S_T) \Leftrightarrow U_t = U_T + c(t-T) - (S_t - S_T)$
- $T \leq t \Rightarrow N_t - N_T \perp\!\!\!\perp N_T$ (notice the intervals $[t, T[,]T, 0]$ are disjoint) $\Rightarrow U_T \perp\!\!\!\perp S_t - S_T$
- $N_t - N_T \sim N_{t-T}$ (Homogenous Poisson Process)

First Approach

Proceding as before (and as the Wald's identity proof) now conditioning on (N_T, N_t) we get:

$$\begin{aligned}
\mathbb{E}[e^{-rU_t} \mid T \leq t] &= e^{-rc(t-T)}\,\mathbb{E}[e^{-rU_T} e^{r(S_t - S_T)} \mid T \leq t] \\
&= e^{-rc(t-T)}\,\mathbb{E}[e^{-rU_T} \mid T \leq t]\,\mathbb{E}[e^{r(S_t - S_T)} \mid T \leq t] \\
&= e^{-rc(t-T)}\,\mathbb{E}[e^{-rU_T} \mid T \leq t]\,\mathbb{E}[m_X(r)^{N_t - N_T} \mid T \leq t] \\
&= e^{-rc(t-T)}\,\mathbb{E}[e^{-rU_T} \mid T \leq t]\,\mathbb{E}[m_X(r)^{N_{t-T}} \mid T \leq t] \\
&= e^{-rc(t-T)}\,\mathbb{E}[e^{-rU_T} \mid T \leq t] e^{\lambda(t-T)(m_X(r)-1)} \\
&= \exp[-rc(t-T) + \lambda(t-T)(m_X(r)-1)] \times \\
&\times \mathbb{E}[e^{-rU_T} \mid T \leq t]
\end{aligned}$$

$$\Rightarrow \mathbb{E}[e^{-RU_t} \mid T \leq t] = \mathbb{E}[e^{-RU_T} \mid T \leq t]$$

First Approach

Up to now, we have:

$$e^{-Ru} = \mathbb{E}[e^{-RU_T} \mid T \leq t]\,\mathbb{P}(T \leq t) + \lim_{t\to\infty} \mathbb{E}[e^{-RU_t} \mid T > t]\,\mathbb{P}(T > t)$$

First Approach

If we can show that $\lim_{t\to\infty} \mathbb{E}[e^{-rU_t} \mid T > t]\,\mathbb{P}(T > t) = 0$ we might have something useful, since we can express the *propability of ruin* as a function of the other terms, which are possible to estimate.

Notice that $T > t \Rightarrow U_t \geq 0$.

With some abuse of notation:

First Approach

$$
\begin{aligned}
&\mathbb{E}[e^{-RU_t} \mid T > t]\,\mathbb{P}(T > t) = \mathbb{E}[e^{-RU_t}, T > t] \\
&= \mathbb{E}[e^{-RU_t}, T > t, 0 \geq U_t \leq b_t] + \mathbb{E}[e^{-RU_t}, T > t, U_t > b_t] \\
&= \mathbb{E}[e^{-RU_t} \mid T > t, 0 \geq U_t \leq b_t]\,\mathbb{P}(T > t, 0 \geq U_t \leq b_t) + \mathbb{E}[e^{-RU_t} \mid T > t, \\
&\leq \mathbb{P}(T > t, 0 \geq U_t \leq b_t) + \mathbb{E}[e^{-RU_t} \mid T > t, U_t > b_t]\,\mathbb{P}(T > t, U_t > b_t) \\
&\leq \mathbb{P}(U_t \leq b_t) + \mathbb{E}[e^{-RU_t} \mid T > t, U_t > b_t]\,\mathbb{P}(T > t, U_t > b_t) \\
&\leq \mathbb{P}(U_t \leq b_t) + \mathbb{E}\, e^{-Rb_t}
\end{aligned}
$$

First Approach

i.e.,

$$\mathbb{E}[e^{-RU_t} \mid T > t]\,\mathbb{P}(T > t) \leq \mathbb{P}(U_t \leq b_t) + \mathbb{E}\,e^{-Rb_t}$$

for some $b_t : b_t \to \infty$ if $t \to \infty$, and we used that $(U_t, R > 0 \wedge U_t > b_t) \Rightarrow$

- $e^{-RU_t} \leq 1 \Rightarrow \mathbb{E}[e^{-RU_t} \mid T > t, 0 \leq U_t \leq b_t] \leq 1$
- $\mathbb{P}(T > t, 0 \leq U_t \leq b_t) = \mathbb{P}(T > t \mid 0 \leq U_t \leq b_t)\,\mathbb{P}(0 \leq U_t \leq b_t)$

  $\leq \mathbb{P}(0 \leq U_t \leq b_t) = \mathbb{P}(0 \leq U_t \bigcap U_t \leq b_t) \leq \mathbb{P}(U_t \leq b_t)$
- $\mathbb{E}[e^{-RU_t} \mid T > t, U_t > b_t] \leq \mathbb{E}\,e^{-Rb_t}$
- $\mathbb{P}(T > t, U_t > b_t) \leq 1$

First Approach

Now we want to show that $\lim_{t\to\infty} \mathbb{P}(U_t \leq b_t) = \lim_{t\to\infty} \mathbb{E}\, e^{-Rb_t} = 0$.
Notice that:

$$\begin{aligned}
&\mathbb{P}(|U_t - \mathbb{E}\, U_t| \geq -(b_t - \mathbb{E}\, U_t)) = \\
&= \mathbb{P}[(U_t - \mathbb{E}\, U_t \leq b_t - \mathbb{E}\, U_t) \cup (U_t - \mathbb{E}\, U_t \geq -(b_t - \mathbb{E}\, U_t))] \\
&\geq \mathbb{P}(U_t - \mathbb{E}\, U_t \leq b_t - \mathbb{E}\, U_t) = \mathbb{P}(U_t \leq b_t)
\end{aligned}$$

First Approach

If we choose b_t such that $\mathbb{E}\, U_t - b_t > 0$ we can apply *Chebychev's inequality* to get:

$$\mathbb{P}(U_t \leq b_t) \leq \mathbb{P}(|U_t - \mathbb{E}\, U_t| \geq \mathbb{E}\, U_t - b_t) \leq \frac{Var(U_t)}{(\mathbb{E}\, U_t - b_t)^2}$$

First Approach

To estimate the variance:

$$
\begin{aligned}
Var(U_t) &= \mathbb{E}\, U_t^2 - \mathbb{E}^2\, U_t \\
\mathbb{E}\, U_t^2 &= \mathbb{E}(u + ct - S_t)^2 = \\
&= \mathbb{E}(u + ct - \mathbb{E}\, S_t)^2 + \mathbb{E}(S_t - \mathbb{E}\, S_t)^2
\end{aligned}
$$

First Approach

We only need to estimate $\mathbb{E}(S_t - \mathbb{E}\, S_t)^2$. By *Wald's identity for the variance* we have:

$$\begin{aligned}\mathbb{E}(S_t - \mathbb{E}\, S_t)^2 =: VarS_t &= \mathbb{E}\, N_t Var(X) + \mathbb{E}^2\, X Var N_t \\ &= \lambda t(VarX + \mathbb{E}^2\, X)\end{aligned}$$

First Approach

$$\mathbb{E}(u + ct - \mathbb{E}\, S_t)^2 = \mathbb{E}^2(u + ct - \mathbb{E}\, S_t) = \mathbb{E}^2(u + ct - S_t) = \mathbb{E}^2\, U_t$$
$$\Rightarrow Var(U_t) = \mathbb{E}\, U_t^2 - \lambda t(VarX + \mathbb{E}^2\, X) - \mathbb{E}^2\, U_t = \lambda t(VarX + \mathbb{E}^2\, X)$$

Choose b_t s.t. $\mathbb{E}\, U_t - b_t = t^k \lambda(VarX + \mathbb{E}^2\, X) > 0$ and we have:

$$\frac{Var(U_t)}{(b_t - \mathbb{E}\, U_t)^2} = \frac{1}{t^{k-1}} \to 0 \quad \forall k > 1$$

We conclude that

$$\lim_{t\to\infty} \mathbb{P}(T \leq t) = \frac{e^{-Ru}}{\lim_{t\to\infty} \mathbb{E}[e^{-RU_T} \mid T \leq t]}$$

Notice that $\lim_{t\to\infty} \mathbb{P}(T \leq t) = \mathbb{P}(T < \infty)$

First Approach

$$\mathbb{P}(T < \infty) = \frac{e^{-Ru}}{\mathbb{E}[e^{-RU_T} \mid T < \infty]}$$

Furthermore, $\mathbb{E}[e^{-RU_T} \mid T < \infty] \geq 1$ since $U_T \leq 0$ and so we have

$$\mathbb{P}(T < \infty) \leq e^{-Ru}$$

Martingale Approach

Computing the *m.g.f* of $U_t - u$, for $r > 0$:

$$\mathbb{E}\, e^{-r(U_t - u)} = e^{-rct}\, \mathbb{E}\, e^{r \sum_{i=0}^{N_t} X_i}$$

Repeating the procedure seen on the "usual" approach, we get:

$$\mathbb{E}\, e^{-r(U_t - u)} = e^{t((m_X(r)-1)\lambda - cr)}$$

Martingale Approach

Define $g(r) = \lambda(m_X(r) - 1) - cr$. Notice that $tg(r) = f(r)$, and they have the same roots (as a function of r), which we will denote again by $R : g(R) = 0$.
As seen in before, but now as a function of g, conditioning on (N_t, N_s): REF

$$s < t \Rightarrow \mathbb{E}\, e^{U_t - U_s} = e^{(t-s)g(r)}.$$

Take the sigma-algebra $\mathscr{F} = \{\mathscr{F}_t\}_{t>0}$ s.t.$\mathscr{F}_t = \sigma(U_s : s \leq t)$
Again, by the independence of increments property of the homogenous poisson process we have that $N_t - N_s \perp\!\!\!\perp \mathscr{F}_s \Rightarrow U_t - U_s \perp\!\!\!\perp \mathscr{F}_s$. Taking inspiration from the last calculations we can construct a martingal:

$$\begin{aligned}
\mathbb{E}[e^{-r(U_t-U_s)-(t-s)g(r)} \mid \mathscr{F}_s] &= e^{-(t-s)g(r)}\,\mathbb{E}[e^{-r(U_t-U_s)} \mid \mathscr{F}_s] \\
&= e^{-(t-s)g(r)}\,\mathbb{E}[e^{-r(U_t-U_s)}] \quad \text{by independence} \\
&= e^{-(t-s)g(r)} e^{(t-s)g(r)} \\
&= 1
\end{aligned}$$

Martingale Approach

That is, $Z_t := e^{-rU_t - tg(r)}$ has the martingale property, and it is in fact a martingale.
Notice that the *time of ruin* is in fact a stopping time, and $T \wedge t$ is also a stopping time $\forall t \geq 0$.
By the *Doob's Optional Stopping Theorem* we have that
$\mathbb{E}\, Z_{T \wedge t} = \mathbb{E}\, Z_0 (= e^{-ru})$.
Therefore, we have that:

Martingale Approach

$$
\begin{aligned}
e^{ru} &= \mathbb{E}\, Z_0 = \mathbb{E}\, Z_{T\wedge t} \\
&= \mathbb{E}[Z_{T\wedge t} \mid T \leq t]\, \mathbb{P}(T \leq t) + \mathbb{E}[Z_{T\wedge t} \mid T > t]\, \mathbb{P}(T > t) \\
&\geq \mathbb{E}[Z_{T\wedge t} \mid T \leq t]\, \mathbb{P}(T \leq t) \\
&= \mathbb{E}[Z_T \mid T \leq t]\, \mathbb{P}(T \leq t) \\
&= \mathbb{E}[e^{-rU_T - Tg(r)} \mid T \leq t]\, \mathbb{P}(T \leq t) \\
&\geq \mathbb{E}[e^{-Tg(r)} \mid T \leq t]\, \mathbb{P}(T \leq t) \\
&\geq \min_{0\leq s\leq t} e^{-sg(r)}\, \mathbb{P}(T \leq t)
\end{aligned}
$$

Martingale Approach

And we get 2 useful inequalities:

$$\mathbb{P}(T \leq t) \leq \frac{e^{-ru}}{\mathbb{E}[e^{-rU_T - Tg(r)} \mid T \leq t]}$$

$$\mathbb{P}(T \leq t) \leq e^{-ru} \max_{0 \leq s \leq t} e^{sg(r)}$$

Martingale Approach

Taking the limit and evaluating at $r = R$, $(g(R) = 0)$ we get the Fundamental Theorem and the Lundberg inequality:

$$\mathbb{P}(T < \infty) \leq \frac{e^{-Ru}}{\mathbb{E}[e^{-RU_T} \mid T < \infty]}$$
$$\mathbb{P}(T < \infty) \leq e^{-Ru}$$

Example

Example

Remember that in order to operate with clear profit we assumed that our model follows $c > \lambda \mathbb{E} X$, thus makes sense to define a *safety coefficient* $\alpha > 0$, through $c = (1 + \alpha)\lambda \mathbb{E} X$.

Take $X \sim Exp(\theta)$. Let T be the time of ruin, b the capital right before the ruin, and $y > 0$.

Example

Notice that $\{0 > y + U_T\} = \{X > b + y \mid X > b\}$: the claim that originated the ruin (the "fall" X) must be larger than our value right before (b), plus some y small enough, *i.e.,*:
$X = \mid U_T \mid + b \Leftrightarrow X > b + y : y > \mid U_T \mid = -U_T$, i.e.,
$\{0 > y + U_T\} = \{X > b + y \mid X > b\}$, thus we can show that
$-U \sim Exp(\theta)$:

Example

$$
\begin{aligned}
\mathbb{P}(-U_T > y \mid T < \infty) =& \mathbb{P}(X > b + y \mid X > b) \\
=& \frac{\mathbb{P}(X > b + y, X > b)}{\mathbb{P}(X > b)} \\
=& \frac{\mathbb{P}(X > b + y)}{\mathbb{P}(X > b)} \\
=& \frac{e^{-\theta(b+y)}}{e^{-\theta b}} = e^{-\theta y} = \mathbb{P}(X > y)
\end{aligned}
$$

And we have that $\mathbb{E}[e^{-RU_T} \mid T < \infty] = \frac{\theta}{\theta - R}, \theta > R$. Let $\alpha = \frac{c}{\lambda\mu} - 1$ be the safety coefficient.

Example

$$\mathbb{P}(T < \infty \mid U_0 = u) = \frac{e^{-Ru}}{\mathbb{E}[e^{-RU_T} \mid T < \infty]} = \frac{e^{-\alpha\theta u/(1+\alpha)}}{\frac{\theta}{\theta - R}}$$
$$= \frac{1}{1+\alpha} e^{-\alpha\theta u/(1+\alpha)}$$

Notice that $\mathbb{P}(T < \infty \mid U_0 = 0) = \frac{1}{1+\alpha} = \frac{\lambda\mu}{c}$

Observations

Some (general) observations related to the Fundamental theorem:

- $\theta \to 0 \Rightarrow R \to 0 \Rightarrow \mathbb{P}(T < \infty \mid U_0 = u) \to 1$
- $\theta \leq 0 \Rightarrow \mathbb{P}(T < \infty \mid U_0 = u) = 1$ (not safe $\Rightarrow$ ruin)
- fixed u, $\lim_{R\to\infty} \mathbb{P}(T < \infty \mid U_0 = u) = 0$
- fixed R, $\lim_{u\to\infty} \mathbb{P}(T < \infty \mid U_0 = u) = 0$ (the larger the initial capital, the smaller the probability of ruin)

THANK YOU!