Martingales in Ruin theory

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Introduction
Introduction

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Useful tools
Chebychev’s inequality:

Let $X$ be a random variable with finite variance, and let $\lambda > 0$. Then:

$$\mathbb{P}[|X - \mathbb{E}X| \geq \lambda] \leq \frac{\text{Var}X}{\lambda^2}$$
(Karr, 1993, p. 91; Kulkarni, 1995, p. 203)
A counting process $\{N_t\}_{t \geq 0}$ is said to be a (homogenous) Poisson Process with rate $\lambda$ if:

- $\{N_t\}_{t \geq 0}$ has independent and stationary increments
- $N_t \sim \text{Poisson}(\lambda t)$
Wald’s identities

For the mean: \( N_t \) r.v. assuming positive integer values, \( X_i \) sequence of i.i.d. r.v.’s, \( X_i \perp \!
\!\!\!\!\!\!\!\!\!\perp N \), then:

\[
\mathbb{E} \sum_{i=1}^{N_t} X_i = \mathbb{E} N_t \mathbb{E} X
\]

For the variance:

\[
\text{Var} \sum_{i=1}^{N_t} X_i = \mathbb{E} N_t \text{Var}X + \mathbb{E}^2 X \text{Var}N_t
\]
Doob’s Optional stopping theorem

Let \((\mathcal{F}_t)_{t \geq 0}\) be a filtration defined on the probability space \((\Omega, \mathcal{F}, P)\), and let \((M_t)_{t \geq 0}\) be a stochastic process adapted to the filtration \((\mathcal{F}_t)_{t \geq 0}\) whose paths are right continuous and locally bounded. The following properties are equivalent:

- \((M_t)_{t \geq 0}\) is a martingale w.r.t. \((\mathcal{F}_t)_{t \geq 0}\)
- For any almost surely bounded stopping time \(T\) of the filtration \((\mathcal{F}_t)_{t \geq 0}\) such that \(\mathbb{E} |M_T| < \infty\) we have \(\mathbb{E} M_T = \mathbb{E} M_0\)
Cramer Lundberg model
The evolution of the capital $U = (U_t)_{t \geq 0}$ of a certain insurance company takes place in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ as follows: The initial capital is $U_0 = u > 0$. Insurance payments arrive continuously at a constant rate $c > 0$ and claims are received at random times $0 < T_1 < T_2 < \ldots$, where the amounts to paid out at these times are described by nonnegative r.v.’s $X_1, X_2, \ldots$. 
Cramer Lundberg model

i.e., the capital $U_t$ at time $t > 0$ is determined by the formula

$$U_t = u + ct - S_t$$

where

- $S_t = \sum_{i=1}^{N_t} X_i$ represents the total amount of claims
- $N_T$ is the number of claims up to time $t$

We will assume that

- $(X_i)_{i \geq 1}$ is an i.i.d sequence of r.v.'s
- $(X_i)_{i \geq 1} \perp N_t \quad \forall t \geq 0$
- $(N_t)_{t \geq 0}$ is a (Poisson) counting process
- $\mathbb{E} X > 0$
One of the main questions relating to the operation of an insurance company is the calculation of the probability of ruin, \( \mathbb{P}(T < \infty) \), and the probability of ruin before time \( t \), \( \mathbb{P}(T \leq t) \).

By Wald’s identity, notice that

\[
\mathbb{E}(U_t - U_0) = ct - \mathbb{E} S_t = ct - \mathbb{E} \sum_{i=1}^{N_t} X_i \\
= ct - \mathbb{E} N_t \mathbb{E} X \\
= t(c - \lambda \mathbb{E} X)
\]
Thus, in the case under consideration, a natural requirement for an insurance company to operate with a clear profit is that $E U_t - U_0 > 0$, i.e., $c > \lambda E X$

We also define the **time of ruin**, $T$:

$$T := \inf\{t \geq 0 : U_t \leq 0\}$$

i.e., the first time at which the insurance company’s capital becomes less than or equal to zero. Of course, $U_t \geq 0 \ \forall t \geq 0 \Rightarrow T = \infty$
Our main objective is to derive an upper bound for the *probability of ruin*, which we shall do with two different approaches: a more intuitive and longer approach, and a martingale approach.
First Approach
Given that we have an expression for the (poisson) point process $N_t$, seems reasonable to try and compute the moment generating function of our process $U_t - mU_t$, which will be a function of the m.g.f of $N_t$, which, since $\{X_i\}_i$ are i.i.d will surely be a function of $m_X$. Conditioning on the time of ruin we can get an expression depending explicitly on $m_X$ and $\psi(u)$, which might prove to be useful.
The trick is to decompose $\mathbb{E} e^{-rU_t}$ in order to get $\mathbb{P}(T \leq t)$:

$$\mathbb{E} e^{-rU_t} = \mathbb{E}[e^{-rU_t} \mid T \leq t] \mathbb{P}(T \leq t) + \mathbb{E}[e^{-rU_t} \mid T > t] \mathbb{P}(T > t)$$  \hspace{1cm} (1)
First Approach

But also simplify $\mathbb{E}[e^{Ut}]$ in order to get $m_X(r)$:

$$\mathbb{E}[e^{-rUt}] = e^{-ru - crt} \mathbb{E} e^{rS_t} = e^{-(ru + crt)} m_{S_t}(r)$$

$$m_{S_t}(r) = \mathbb{E} e^{rS_t} = \mathbb{E} e^{r \sum_{i=0}^{N_t}} = \sum_n \mathbb{E}[e^{r \sum_{i=0}^{N_t} | N_t = n}] \mathbb{P}(N_t = n)$$

$$= \sum_n \mathbb{E}[e^{r \sum_{i=0}^{N_t} | N_t = n}] \frac{(\lambda t)^n}{n!} e^{-\lambda t}$$

$$= \sum_n m_X^n(r) \frac{(\lambda t)^n}{n!} = e^{\lambda t(m_X(r) - 1)}$$

$$\Rightarrow \mathbb{E}[e^{-rUt}] = e^{-(ru + crt)} e^{\lambda t(m_X(r) - 1)}$$

$$= \exp[-ru - crt + \lambda t(m_X(r) - 1)]$$
First Approach

Our life would be much easier if $t = 0$ or if there was

$$r : g(r) = -crt + \lambda t(m_X(r) - 1) = 0.$$ 

In fact, there is an unique solution, which we shall denote by $R$.

Therefore, for such $R$ we get:

$$\mathbb{E}[e^{-Ru}] = e^{-Ru}$$

which is NOT a function of time. Therefore we can take the limit and get:

$$e^{-Ru} = \lim_{t \to \infty} \mathbb{E}[e^{-rU_t} \mid T \leq t] \mathbb{P}(T \leq t) + \lim_{t \to \infty} \mathbb{E}[e^{-rU_t} \mid T > t] \mathbb{P}(T > t)$$
First Approach

- \( U_t - U_T = c(t - T) - (S_t - S_T) \iff U_t = U_T + c(t - T) - (S_t - S_T) \)
- \( T \leq t \Rightarrow N_t - N_T \perp N_T \) (notice the intervals \([t, T], T, 0]\) are disjoint)
  \( \Rightarrow U_T \perp S_t - S_T \)
- \( N_t - N_T \sim N_{t-T} \) (Homogenous Poisson Process)
First Approach

Proceding as before (and as the Wald’s identity proof) now conditioning on \((N_T, N_t)\) we get:

\[
\mathbb{E}[e^{-rU_t} \mid T \leq t] = e^{-rc(t-T)} \mathbb{E}[e^{-rU_T} e^{r(S_t-S_T)} \mid T \leq t]
\]

\[
= e^{-rc(t-T)} \mathbb{E}[e^{-rU_T} \mid T \leq t] \mathbb{E}[e^{r(S_t-S_T)} \mid T \leq t]
\]

\[
= e^{-rc(t-T)} \mathbb{E}[e^{-rU_T} \mid T \leq t] \mathbb{E}[m_X(r)^{N_t-N_T} \mid T \leq t]
\]

\[
= e^{-rc(t-T)} \mathbb{E}[e^{-rU_T} \mid T \leq t] \mathbb{E}[m_X(r)^{N_t-T} \mid T \leq t]
\]

\[
= e^{-rc(t-T)} \mathbb{E}[e^{-rU_T} \mid T \leq t] e^{\lambda(t-T)(m_X(r)-1)}
\]

\[
= \exp[-rc(t-T) + \lambda(t-T)(m_X(r)-1)] \times \mathbb{E}[e^{-rU_T} \mid T \leq t]
\]

\[
\Rightarrow \mathbb{E}[e^{-RU_t} \mid T \leq t] = \mathbb{E}[e^{-RU_T} \mid T \leq t]
\]
First Approach

Up to now, we have:

$$e^{-Ru} = \mathbb{E}[e^{-RU_T} \mid T \leq t] \mathbb{P}(T \leq t) + \lim_{t \to \infty} \mathbb{E}[e^{-Ru_T} \mid T > t] \mathbb{P}(T > t)$$
If we can show that \( \lim_{t \to \infty} \mathbb{E}[e^{-rU_t} \mid T > t] \mathbb{P}(T > t) = 0 \) we might have something useful, since we can express the probability of ruin as a function of the other terms, which are possible to estimate.

Notice that \( T > t \Rightarrow U_t \geq 0 \).

With some abuse of notation:
First Approach

\[
\mathbb{E}[e^{-RU_t} \mid T > t] \mathbb{P}(T > t) = \mathbb{E}[e^{-RU_t}, T > t] \\
= \mathbb{E}[e^{-RU_t}, T > t, 0 \geq U_t \leq b_t] + \mathbb{E}[e^{-RU_t}, T > t, U_t > b_t] \\
= \mathbb{E}[e^{-RU_t} \mid T > t, 0 \geq U_t \leq b_t] \mathbb{P}(T > t, 0 \geq U_t \leq b_t) + \mathbb{E}[e^{-RU_t} \mid T > t, U_t > b_t] \\
\leq \mathbb{P}(T > t, 0 \geq U_t \leq b_t) + \mathbb{E}[e^{-RU_t} \mid T > t, U_t > b_t] \mathbb{P}(T > t, U_t > b_t) \\
\leq \mathbb{P}(U_t \leq b_t) + \mathbb{E}[e^{-RU_t} \mid T > t, U_t > b_t] \mathbb{P}(T > t, U_t > b_t) \\
\leq \mathbb{P}(U_t \leq b_t) + \mathbb{E} e^{-Rb_t}
\]
First Approach

\[ \mathbb{E}[e^{-RU_t} \mid T > t] \mathbb{P}(T > t) \leq \mathbb{P}(U_t \leq b_t) + \mathbb{E} e^{-Rb_t} \]

for some \( b_t : b_t \to \infty \) if \( t \to \infty \), and we used that

\( (U_t, R > 0 \land U_t > b_t) \Rightarrow \)

\( e^{-RU_t} \leq 1 \Rightarrow \mathbb{E}[e^{-RU_t} \mid T > t, 0 \leq U_t \leq b_t] \leq 1 \)

\( \mathbb{P}(T > t, 0 \leq U_t \leq b_t) = \mathbb{P}(T > t \mid 0 \leq U_t \leq b_t) \mathbb{P}(0 \leq U_t \leq b_t) \)

\( \leq \mathbb{P}(0 \leq U_t \leq b_t) = \mathbb{P}(0 \leq U_t \cap U_t \leq b_t) \leq \mathbb{P}(U_t \leq b_t) \)

\( \mathbb{E}[e^{-RU_t} \mid T > t, U_t > b_t] \leq \mathbb{E} e^{-Rb_t} \)

\( \mathbb{P}(T > t, U_t > b_t) \leq 1 \)
First Approach

Now we want to show that $\lim_{t \to \infty} \mathbb{P}(U_t \leq b_t) = \lim_{t \to \infty} \mathbb{E} e^{-Rb_t} = 0$. Notice that:

\[
\mathbb{P}(|U_t - \mathbb{E} U_t| \geq -(b_t - \mathbb{E} U_t)) = \\
= \mathbb{P}([U_t - \mathbb{E} U_t \leq b_t - \mathbb{E} U_t] \cup [U_t - \mathbb{E} U_t \geq -(b_t - \mathbb{E} U_t)]) \\
\geq \mathbb{P}(U_t - \mathbb{E} U_t \leq b_t - \mathbb{E} U_t) = \mathbb{P}(U_t \leq b_t)
\]
First Approach

If we choose $b_t$ such that $\mathbb{E} U_t - b_t > 0$ we can apply Chebychev’s inequality to get:

$$P(U_t \leq b_t) \leq P(|U_t - \mathbb{E} U_t| \geq \mathbb{E} U_t - b_t) \leq \frac{\text{Var}(U_t)}{(\mathbb{E} U_t - b_t)^2}$$
To estimate the variance:

\[ \text{Var}(U_t) = \mathbb{E} U_t^2 - \mathbb{E}^2 U_t \]

\[ \mathbb{E} U_t^2 = \mathbb{E} (u + ct - S_t)^2 = \]

\[ = \mathbb{E} (u + ct - \mathbb{E} S_t)^2 + \mathbb{E} (S_t - \mathbb{E} S_t)^2 \]
We only need to estimate $\mathbb{E}(S_t - \mathbb{E} S_t)^2$. By Wald's identity for the variance we have:

$$
\mathbb{E}(S_t - \mathbb{E} S_t)^2 =: \text{Var}S_t = \mathbb{E} N_t \text{Var}(X) + \mathbb{E}^2 X \text{Var}N_t
$$

$$
= \lambda t (\text{Var}X + \mathbb{E}^2 X)
$$
First Approach

\[ \mathbb{E}(u + ct - \mathbb{E} S_t)^2 = \mathbb{E}^2(u + ct - \mathbb{E} S_t) = \mathbb{E}^2(u + ct - S_t) = \mathbb{E}^2 U_t \]

\[ \Rightarrow \text{Var}(U_t) = \mathbb{E} U_t^2 - \lambda t(\text{Var} \mathcal{X} + \mathbb{E}^2 \mathcal{X}) - \mathbb{E}^2 U_t = \lambda t(\text{Var} \mathcal{X} + \mathbb{E}^2 \mathcal{X}) \]

Choose \( b_t \) s.t. \( \mathbb{E} U_t - b_t = t^k \lambda (\text{Var} \mathcal{X} + \mathbb{E}^2 \mathcal{X}) > 0 \) and we have:

\[ \frac{\text{Var}(U_t)}{(b_t - \mathbb{E} U_t)^2} = \frac{1}{t^{k-1}} \to 0 \quad \forall k > 1 \]
First Approach

We conclude that

\[
\lim_{t \to \infty} \mathbb{P}(T \leq t) = \frac{e^{-Ru}}{\lim_{t \to \infty} \mathbb{E}[e^{-RU_T} \mid T \leq t]}
\]

Notice that \( \lim_{t \to \infty} \mathbb{P}(T \leq t) = \mathbb{P}(T < \infty) \)
First Approach

\[ \mathbb{P}(T < \infty) = \frac{e^{-Ru}}{\mathbb{E}[e^{-RU_T} \mid T < \infty]} \]

Furthermore, \( \mathbb{E}[e^{-RU_T} \mid T < \infty] \geq 1 \) since \( U_T \leq 0 \) and so we have

\[ \mathbb{P}(T < \infty) \leq e^{-Ru} \]
Martingale Approach
Martingale Approach

Computing the $m.g.f$ of $U_t - u$, for $r > 0$:

\[ \mathbb{E} e^{-r(U_t-u)} = e^{-rct} \mathbb{E} e^{r \sum_{i=0}^{N_t} X_i} \]

Repeating the procedure seen on the "usual" approach, we get:

\[ \mathbb{E} e^{-r(U_t-u)} = e^{t((m_X(r)-1)\lambda-cr)} \]
Define $g(r) = \lambda(m_X(r) - 1) - cr$. Notice that $tg(r) = f(r)$, and they have the same roots (as a function of $r$), which we will denote again by $R : g(R) = 0$.

As seen in before, but now as a function of $g$, conditioning on $(N_t, N_s)$:

$$s < t \Rightarrow \mathbb{E} e^{U_t - U_s} = e^{(t-s)g(r)}.$$
Martingale Approach

Take the sigma-algebra $\mathcal{F} = \{\mathcal{F}_t\}_{t>0}$ s.t. $\mathcal{F}_t = \sigma(U_s : s \leq t)$

Again, by the independence of increments property of the homogeneous poisson process we have that $N_t - N_s \perp\mathcal{F}_s \Rightarrow U_t - U_s \perp\mathcal{F}_s$. Taking inspiration from the last calculations we can construct a martingal:

$$
\mathbb{E}[e^{-r(U_t-U_s)-(t-s)g(r)} \mid \mathcal{F}_s] = e^{-(t-s)g(r)} \mathbb{E}[e^{-r(U_t-U_s)} \mid \mathcal{F}_s] \\
= e^{-(t-s)g(r)} \mathbb{E}[e^{-r(U_t-U_s)}] \quad \text{by independence} \\
= e^{-(t-s)g(r)} e^{(t-s)g(r)} \\
= 1
$$
Martingale Approach

That is, \( Z_t := e^{-rU_t - t g(r)} \) has the martingale property, and it is in fact a martingale.
Notice that the time of ruin is in fact a stopping time, and \( T \land t \) is also a stopping time \( \forall t \geq 0 \).
By the Doob’s Optional Stopping Theorem we have that
\[
\mathbb{E} Z_{T \land t} = \mathbb{E} Z_0 (= e^{-ru}).
\]
Therefore, we have that:
Martingale Approach

\[ e^{ru} = \mathbb{E} Z_0 = \mathbb{E} \left[ Z_{T \wedge t} \right] \]

\[ = \mathbb{E} \left[ Z_{T \wedge t} \mid T \leq t \right] \mathbb{P}(T \leq t) + \mathbb{E} \left[ Z_{T \wedge t} \mid T > t \right] \mathbb{P}(T > t) \]

\[ \geq \mathbb{E} \left[ Z_{T \wedge t} \mid T \leq t \right] \mathbb{P}(T \leq t) \]

\[ = \mathbb{E} \left[ Z_T \mid T \leq t \right] \mathbb{P}(T \leq t) \]

\[ = \mathbb{E} \left[ e^{-rU_T - Tg(r)} \mid T \leq t \right] \mathbb{P}(T \leq t) \]

\[ \geq \mathbb{E} \left[ e^{-Tg(r)} \mid T \leq t \right] \mathbb{P}(T \leq t) \]

\[ \geq \min_{0 \leq s \leq t} e^{-sg(r)} \mathbb{P}(T \leq t) \]
Martingale Approach

And we get 2 useful inequalities:

\[ \mathbb{P}(T \leq t) \leq \frac{e^{-ru}}{\mathbb{E}[e^{-rU_T - Tg(r)} \mid T \leq t]} \]

\[ \mathbb{P}(T \leq t) \leq e^{-ru} \max_{0 \leq s \leq t} e^{sg(r)} \]
Taking the limit and evaluating at $r = R$, $(g(R) = 0)$ we get the Fundamental Theorem and the Lundberg inequality:

$$\mathbb{P}(T < \infty) \leq \frac{e^{-Ru}}{\mathbb{E}[e^{-RU_T} \mid T < \infty]}$$

$$\mathbb{P}(T < \infty) \leq e^{-Ru}$$
Example
Example

Remember that in order to operate with clear profit we assumed that our model follows $c > \lambda \mathbb{E} X$, thus makes sense to define a safety coefficient $\alpha > 0$, through $c = (1 + \alpha) \lambda \mathbb{E} X$.

Take $X \sim \text{Exp}(\theta)$. Let $T$ be the time of ruin, $b$ the capital right before the ruin, and $y > 0$. 
Example

Notice that \( \{0 > y + U_T\} = \{X > b + y \mid X > b\} \): the claim that originated the ruin (the "fall" \( X \)) must be larger than our value right before \( (b) \), plus some \( y \) small enough, i.e.,:

\[
X = \mid U_T \mid + b \iff X > b + y : y > \mid U_T \mid = -U_T, \text{i.e.,}
\]

\[
\{0 > y + U_T\} = \{X > b + y \mid X > b\}, \text{thus we can show that}
\]

\[-U \sim \text{Exp}(\theta):\]
Example

\[\mathbb{P}(-U_T > y \mid T < \infty) = \mathbb{P}(X > b + y \mid X > b)\]
\[= \frac{\mathbb{P}(X > b + y, X > b)}{\mathbb{P}(X > b)}\]
\[= \frac{\mathbb{P}(X > b + y)}{\mathbb{P}(X > b)}\]
\[= e^{-\theta(b+y)} / e^{-\theta y} = e^{-\theta y} = \mathbb{P}(X > y)\]

And we have that \(\mathbb{E}[e^{-RU_T} \mid T < \infty] = \frac{\theta}{\theta - R}, \theta > R\). Let \(\alpha = \frac{c}{\lambda\mu} - 1\) be the safety coefficient.
Example

\[ \mathbb{P}(T < \infty \mid U_0 = u) = \frac{e^{-Ru}}{\mathbb{E}[e^{-RU_T} \mid T < \infty]} = \frac{e^{-\alpha\theta u/(1+\alpha)}}{\frac{\theta}{\theta - R}} \]

\[ = \frac{1}{1 + \alpha} e^{-\alpha\theta u/(1+\alpha)} \]

Notice that \( \mathbb{P}(T < \infty \mid U_0 = 0) = \frac{1}{1 + \alpha} = \frac{\lambda\mu}{c} \)
Some (general) observations related to the Fundamental theorem:

- \( \theta \to 0 \Rightarrow R \to 0 \Rightarrow \mathbb{P}(T < \infty \mid U_0 = u) \to 1 \)
- \( \theta \leq 0 \Rightarrow \mathbb{P}(T < \infty \mid U_0 = u) = 1 \) (not safe \( \Rightarrow \) ruin)
- fixed \( u \), \( \lim_{R \to \infty} \mathbb{P}(T < \infty \mid U_0 = u) = 0 \)
- fixed \( R \), \( \lim_{u \to \infty} \mathbb{P}(T < \infty \mid U_0 = u) = 0 \) (the larger the initial capital, the smaller the probability of ruin)
THANK YOU!