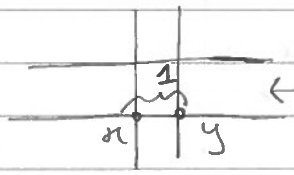
LECTURE 1

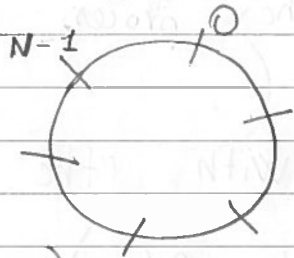
Chapter 1 KL

We consider indistinguishable particles moving as independent RW on $\mathbb{T}_N^d := \mathbb{Z}^d / N\mathbb{Z}^d$

d-dimensional discrete torus of mesh 1



with N along each direction.



We have K particles at time 0 at $x_1, \dots, x_K$ (initial positions)

Independent traslational invariant continuous time RW

$$p(x,y) = p(0, y-x) := p(y-x) \quad \text{on } \mathbb{Z}^d$$

[Skeleton of the Markov chain]

$$\rightsquigarrow \sum_y p(y,x) = 1$$

bistochastic matrix

Remark: $p_t(x,y) = P_x(X(t)=y)$ is still translational invariant

Proof: [discrete time] : $p_t(x,y) = \sum_{k=0}^{+\infty} e^{-t} \frac{t^k}{k!} p^{*k}(x,y)$,

$$p^{*k}(x) = \sum_{x_i} p(x, x_1) \dots p(x_{k-1}, x_k = y)$$

from $p(x_i, x_{i+1}) = p(x_i - x, x_{i+1} - x) = p(0, x_{i+1} - x)$

$$= \sum_{x_i} p(0, x_1 - x)\, p(x_1 - x, x_2 - x_1) \dots p(x_{k-1} - x, y - x)$$

$$= \sum_{y_i} p(0, y_1)\, p(y_1, y_2) \dots p(y_{k-1}, y - x) = p_k(0, y - x)$$ ∎

$\underline{\eta} = \{\eta(x)\}_x$ is the particles configuration,

$\underline{\eta} \in \Sigma_N = \underset{\uparrow}{\mathbb{N}}^{\mathbb{T}_N^d}$ space state of configurations.

single particle state space

The process has two descriptions:

I) With the Markov generator

$$\mathcal{L}_N \underset{\uparrow \text{local}}{f(\eta)} := \sum_{x \in \mathbb{T}_N^d} \sum_{z \in \mathbb{T}_N^d} \underbrace{p^N(z)\, \eta(x)}_{\text{rates of the M.C.}} \left[f(\eta^{x,x+z}) - f(\eta) \right]$$

$$p^N(z) := \sum_{z \in \mathbb{Z}^d} p(x + Nz), \quad \eta^{x,x+z}(y) := \begin{cases} \eta(y), & y \neq x, x+z \\ \eta(x) - 1, & y = x \\ \eta(x+1) + 1, & y = x + 1 \end{cases}$$

This dynamics is irreducible on the hyperplane $\Sigma_N^K := \left\{ \underline{\eta} \in \mathbb{N}^{\mathbb{T}_N^d} \;\middle|\; \sum_{x \in \mathbb{T}_N^d} \eta(x) = K \right\}$

$\boxed{\text{Def.}}$ (Poisson distribution) p_k on $\mathbb{N}$ s.t.

$$p_k = e^{-\alpha} \frac{\alpha^k}{k!} \qquad \left[\langle X \rangle = \alpha, \quad \sigma_X^2 = \alpha \right]$$

$\boxed{\text{Def.}}$ (Laplace transform) $(\lambda > 0)$

$$e^{-\alpha} \sum_{k=0}^{+\infty} e^{-\lambda k} \frac{\alpha^k}{k!} = e^{\alpha(e^{-\lambda}-1)}$$

II) description: K copie of IRWs Z_t^i with label i,

$$X_t^i = x^i + \underbrace{Z_t^i}_{\substack{\uparrow \\ \text{RW from origin } 0}} \mod N$$

X_t^i has transition probality $p_t^N(x,y) := \sum_{z \in \mathbb{Z}^d} p_t(x, y + Nz)$

$\eta_t(x) = \sum_{i=1}^{K} \mathbb{1}\{X_t^i = x\}$ ← I count particles in x at time t.

$\boxed{\text{Def.}}$ (Poisson measure on $\mathbb{T}_N^d$)

Given $\rho : \mathbb{T}_N^d \to \mathbb{R}_+$, it is a probability on Σ_N s.t. 1) $\{\eta(x)\}_x$ are independent

2) $\eta(x)$ is distributed with Poisson distribution of parametre $\rho(x)$.

When $\rho(\cdot)$ is a constant α we denote $\nu_{\rho(\cdot)}^N := \nu_\alpha^N$

$\boxed{\text{Fact}}$ (Feller II volume)

A distribution function F of a pr. measure is uniquely determined by its Laplace transform.

④

~> A Poisson measure is characterized by knowing the Laplace transform $\forall$ finite collection $\{\lambda(x) : x \in \mathbb{T}_N^d\} \geq 0$

$$\mathbb{E}_{\nu_{\rho(\cdot)}^N}\left[\exp\left\{-\sum_{x\in\mathbb{T}_N^d}\eta(x)\lambda(x)\right\}\right] = \exp\sum_{x\in\mathbb{T}_N^d}\rho(x)\left(e^{-\lambda(x)}-1\right)$$

$\boxed{\text{Prop.}}$ If at time 0 I start with a Poisson measure $\nu^N_{\rho(\cdot)}$ then at time $t>0$ I still have a Poisson measure of parametre $\Psi_{N,t}(x) = \sum_{y\in\mathbb{T}_N^d}\rho_0(y/N)\,p^N_t(y,x)$ in particular ν^N_α measure of constant profile are invariant.

$\boxed{\text{Proof}}$

$$\underbrace{\mathbb{E}_{\nu_{\rho_0(\cdot)}}\left[\exp - \sum_{x\in\mathbb{T}_N^d}\lambda(x)\eta_t(x)\right]}_{*} = \exp\left\{\sum_{x\in\mathbb{T}_N^d}\rho_0\left(\frac{x}{N}\right)\left(e^{-\lambda(x)}-1\right)\right\}$$

[respect to $\mathbb{P}_{\nu^N_\alpha}$ pr. measure on $\mathcal{D}(\mathbb{R}_+, \Sigma_N)$ induced by the process and initial measure ν^N_α]

Call $X_t^{y,k}$ position at time t of the k-th particle initially at y.

$$\eta_t(x) = \sum_{y\in\mathbb{T}_N^d}\sum_{k=1}^{\eta_0(y)}\mathbb{1}\left\{X_t^{y,k} = x\right\} \leftarrow$$ [I count how many particles are at x at time t]

$$\sum_x \lambda(x)\eta_t(x) = \sum_{x\in\mathbb{T}_N^d}\lambda(x)\sum_{y\in\mathbb{T}_N^d}\sum_{k=1}^{\eta_0(y)}\mathbb{1}\{X_t^{y,k}=x\}$$

$$= \sum_{y\in\mathbb{T}_N^d}\sum_{k=1}^{\eta_0(y)}\lambda\left(X_t^{y,k}\right)$$

$$\rightsquigarrow * = \mathbb{E}_{\nu^N_{\rho_0(\cdot)}}\left[\exp - \sum_{y\in\mathbb{T}_N^d}\sum_{k=1}^{\eta_0(y)}\lambda\left(X_t^{y,k}\right)\right] =$$

from independence of Poisson ν^N_α for η_0
+
independence of particles

$$= \prod_{y\in\mathbb{T}_N^d} \int \nu^N_{\rho_0(y)}(d\eta) \left(E\left[\exp\left(-\lambda(X_t^y)\right)\right]\right)^{\eta_0(y)}$$

$$= \prod_{y\in\mathbb{T}_N^d} \int \nu^N_{\rho_0(y)}(d\eta) \quad \exp\left(\eta_0(y) - \left(-\log E\left[\exp\left(-\lambda(X_t^y)\right)\right]\right)\right)$$

Laplace transform Poisson

$$\xrightarrow{} = \prod_{y\in\mathbb{T}_N^d} \exp\left[\rho_0\left(\frac{y}{N}\right)\left(e^{\log\left[E\left(\exp\left(-\lambda(X_t^y)\right)\right)\right]} - 1\right)\right]$$

$$= \prod_{y\in\mathbb{T}_N^d} \exp\left[\rho_0\left(\frac{y}{N}\right)\left[E\left(\exp\left(-\lambda(X_t^y)\right)\right) - 1\right]\right]$$

$$= \prod_{y\in\mathbb{T}_N^d} \exp\left\{\rho_0\left(\frac{y}{N}\right)\left[\sum_x p_t^N(x-y)\exp\left(-\lambda(x)\right) - 1\right]\right\}$$

$$= \exp\left\{\sum_{y\in\mathbb{T}_N^d} \rho_0\left(\frac{y}{N}\right)\left[\sum_x p_t^N(x-y)\, e^{-\lambda(x)} - 1\right]\right\}$$

use bistochasticity

$$= \exp \sum_{x\in\mathbb{T}_N^d} \Psi_{N,t}(x)\left(e^{-\lambda(x)} - 1\right)$$

$$\Psi_{N,t}(x) = \sum_y \rho_0\left(\frac{y}{N}\right) p_t^N(x-y) \;\; [p_t^N(y,x)], \quad \text{if } \rho_0\left(\frac{y}{N}\right) = \text{const} = \alpha$$

$$\Psi_{N,t}(x) = \alpha \sum_y p_t^N(x-y) = \alpha \leftarrow$$ invariance of constant Poisson measure.

Remarks

1) Poisson measure are naturally parametrized by the density of particle $\langle \eta(x)\rangle_{\nu^N_{\rho(\cdot)}} = \rho(x)$

and LLN: $\mathbb{P}_{\nu_\alpha^N}\left(\frac{1}{|\mathbb{T}_N^d|}\sum_{x\in\mathbb{T}_N^d} \eta(x) - \alpha\right) \xrightarrow[N\to\infty]{} 0$

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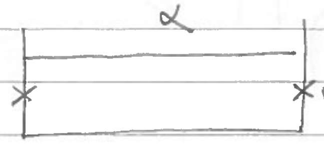

torus with constant stationary profile.

2) For $N < +\infty$, systems is <u>not irreducibile</u> on Σ_N

It looks more natural to consider the invariant measures:

$$\nu_{\mathbb{T}_N^d, K}(\cdot) := \nu_\alpha^N \Big(\cdot \;\Big|\; \underbrace{\sum_{x \in \mathbb{T}_N^d} \eta(x) = K}_{\text{constraint on total number of particles}} \Big)$$

$\nu_{\mathbb{T}_N^d, K}$: canonical measures

ν_α^N: grand-canonical measures

$\boxed{\text{Prop.}}$ The ensembles canonical and grandcanonical are equivalent, i.e.

$$\lim_{N \to \infty} \nu_\alpha^N \Big(\eta(x_1) = k_1, \ldots, \eta(x_r) = k_r \;\Big|\; \sum_{x \in \mathbb{T}_N^d} \eta(x) = [N^d \beta] \Big)$$

$$=$$

$$\nu_\beta \big(\eta(x_1) = k_1, \ldots, \eta(x_r) = k_r \big)$$

$[N^d \beta]$: integer part

<u>Proof</u> we use $P(A \mid B) = P(A, B) / P(B)$

where $A = \left\{ \begin{array}{l} \text{having in } x_i \in \mathbb{T}_N^d \ k_i \text{ particles } i = 1, \ldots, r \\ \text{distributed with Poisson } \alpha \end{array} \right\}$

$B = \left\{ \begin{array}{l} \text{having in } \mathbb{T}_N^d \ N^d \beta \text{ particles} \\ \text{distributed with Poisson } \beta \end{array} \right\}$

Using $\text{Poisson}(\lambda_1) + \text{Poisson}(\lambda_2) = \text{Poisson}(\lambda_1 + \lambda_2)$

$$\frac{\alpha^{k_1+\dots+k_r}}{k_1!\dots k_r!} e^{-\alpha r} \frac{\left[\left(N^d - r\right)\alpha\right]^{N^d\beta - (k_1+\dots k_r)} e^{-\alpha(N^d - r)}}{\left[N^d\beta - (k_1+\dots+k_r)\right]!} \cdot$$

$$\cdot \left[e^{-N^d\alpha} \frac{(\alpha N^d)^{N^d\beta}}{(N^d\beta)!} \right]^{-1} =$$

$$= \frac{\left(N^d - r\right)^{(N^d\beta - (k_1+\dots+k_r))}}{k_1!\dots k_r!} \frac{N^d\beta (N^d\beta - 1)\dots (N^d\beta - (k_1+\dots+k_r)+1)}{\left(N^d\right)^{N^d\beta}}$$

$$= \lim_N \frac{1}{k_1!\dots k_r!} \left[\left(1 - \frac{r}{N^d}\right)^{N^d} \right]^{\beta} \beta^{k_1+\dots+k_r} =$$

$$= \frac{\beta^{k_1+\dots+k_r}}{k_1!\dots k_r!} e^{-r\beta} \quad \left[\exp n = \lim_N \left(1 + \frac{n}{N}\right)^N \right]$$

Microscopic vs Macroscopic space

Let $\mathbb{T}^d \supset \mathbb{T}^d_{1/N} := \frac{\mathbb{Z}^d}{N} / \mathbb{Z}^d$

has vertices $n = y/N$, $y \in \mathbb{T}^d_N$

$|y_1 - y_2| = 1 \longrightarrow |x_1 - x_2| = 1/N$

$\mathbb{T}^d_\varepsilon$ macroscopic space / $\mathbb{T}^d_N$ microscopic space

$\varepsilon := \frac{1}{N}$

two space scale

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Def. [Product measures with slowly varying parametre associated to $\rho : \mathbb{T}^d \to \mathbb{R}_+$]

It is a Poisson measure on $\mathbb{T}_N^d$ associated to a smooth Function $\rho_0 : \mathbb{T}^d \to \mathbb{R}_+$.

$$\left[\underset{\mathbb{T}_N^d}{\rho_0^M(y)} := \rho_0\left(\underset{\mathbb{T}_\varepsilon^d}{\frac{y}{N}}\right) = \rho_0(\underset{\mathbb{T}^d}{u}) \right]$$

Def. $\eta \in \mathbb{T}_N^d$, $(\tau_x \eta)(y) := \eta(y+x)$, $x, y \in \mathbb{T}_N^d$

$\rightsquigarrow (\tau_{-1}\eta)(1) = \eta(1-1)$ [translation to right], $(\tau_x f)(\eta) := f(\tau_x \eta)$

this defines the translations on measure:

$$\int d\mu \; (\tau_x f)(\eta) = \int f(\tau_x \eta)\, d\mu(\eta)$$

$$= \int d\mu(\tau_{-x}\eta)\, f(\eta) = \int d(\tau_x \mu)(\eta)\, f(\eta)$$

change of variable $\tau_x \eta = \eta'$

Limit of a sequence of measures $\{\mu_N\}_N$ on $\mathbb{T}_N^d$:

Consider $\{\mu_N\}_N$, to do a limit in N we embed $\Sigma_N := \mathbb{N}^{\mathbb{T}_N^d}$ in $\tilde{\Sigma}_N := \mathbb{N}^{\mathbb{Z}^d}$ [with product topology] and make periodic a configuration of $\mathbb{T}_N^d$ on $\mathbb{Z}^d$ considering an extend product meas. $\tilde{\mu}_N$ on $\mathbb{N}^{\mathbb{Z}^d}$.

[Remark: this is how to have a well defined weak convergence of $\mu_N \xrightarrow{w} \mu$.]

$\boxed{\text{Prop}}$ On $\mathcal{M}_+^1(\tilde{\Sigma}_N)$ sp. of prob. measures with week topology

$$\int_{\tilde{\Sigma}_N} f \, d\mu^N \xrightarrow[N]{w} \int_{\tilde{\Sigma}_N} f \, d\mu \quad , \forall f \in \mathcal{C}^b(\tilde{\Sigma}_N)$$

$$\Updownarrow$$

$$\int_{\tilde{\Sigma}_N} \psi \, d\mu^N \xrightarrow[N]{w} \int_{\tilde{\Sigma}_N} \psi \, d\mu$$

$\forall$ ψ bounded cylindrical function [dependence on the configuration only through a Finete set of coordinates]

Proof : see Lecture 2.

Meaning of measure with slowing varying parameter:

$$\underbrace{\lim_{N\to\infty} \mathbb{E}_{\nu^N_{\rho_0(\cdot)}} \left[e^{-\sum_{|x|\le \ell} \lambda(x)\, \eta([uN]+x)} \right]}_{\circledast}$$

$$= \mathbb{E}_{\nu_{\rho_0(u)}} \left[e^{-\sum_{|x|\le \ell} \lambda(x)\, \eta(x)} \right] \leftarrow \left[\begin{array}{c} \text{Laplace transform of} \\ \text{Poisson of constant} \\ \text{profile } \rho_0(u) \end{array} \right]$$

because $\displaystyle \lim_{N\to\infty} \rho_0^N([uN]+x) = \lim_{N\to\infty} \rho_0\left(\frac{[uN]+x}{N}\right) = \rho_0(u)$

for $|x| \le \ell$

$$\circledast = \mathbb{E}_{\tau_{[uN]}\nu^N_{\rho_0(\cdot)}} \left[e^{-\sum_{|x|\le \ell} \lambda(x)\, \eta(x)} \right]$$

$\rightsquigarrow$ this is the week convergence

$$\tau_{[uN]}\, \nu^N_{\rho_0(\cdot)} \xrightarrow[N]{(w)} \nu_{\rho_0(u)}$$

i.e. looking around a point $[uN]$/"close" to a point u in a finite box of size $\ell < +\infty$, I see a Poisson measure of constant parameter $\rho_0(\cdot)$.

This is a very strong local equilibrium

$\boxed{\text{Def.}}$ (local equilibrium)

$(\mu^N)_{N\geq 1}$ on $\mathbb{N}^{\mathbb{T}^d_N}$ is a local equilibrium of profile $\rho_0 : \mathbb{T}^d \to \mathbb{R}_+$ if

$\lim\limits_{N\to\infty} \tau_{[uN]}\mu^N \overset{(w)}{=} \nu_{\rho_0(u)}$, $\forall u$ continuity point of $\rho_0(\cdot)$.

Remark: This is a stronger notion of the usual local equilibrium you do in replacement lemma.

$\boxed{\text{Hydrodynamics}}$ We have characterized Poisson measures in terms of finite Laplace transform

$\mathbb{E}_{\nu^N_{\rho(\cdot)}}[\exp - (\sum_{|x|\leq \ell} \lambda(x)\eta_t(x))]$

Starting from $\nu^N_{\rho_0(\cdot)}$ we have at time $t>0$ a Poisson $\nu^N_{\psi_{N,t}(\cdot)}$ with parameter $\psi_{N,t}(x) = \sum_{y\in\mathbb{T}^d_N} \rho_0\left(\frac{y}{N}\right) p^N_t(y,x)$

and therefore a new local equilibrium

$\lim\limits_{N\to\infty} \tau_{[uN]}\nu^N_{\psi_{N,t}(\cdot)} \overset{(w)}{=} \nu_{\rho_t(u)}$ where $\rho_t(u) = \lim\limits_N \psi_{N,t}([uN])$.

$\rightsquigarrow$ $\rho_t(u)$ will be characterized by hydrodynamics.

$\boxed{\text{Th.}}$ [Billingsley, "Convergence of Probability measures"]

$\mathbb{P}$ probability measure on Ω separable complete metric space $\Rightarrow$ every $\mathbb{P}$ probability on $(\Omega, \mathcal{B})$ is tight. [$\mathcal{B}$ is borel σ-algebra].

$\rightsquigarrow$ we have <u>tightness</u> of $p_t(\cdot,\cdot)$ $\left[\sum_{y\in\mathbb{Z}^d} p_t(x,y)=1\right]$

$\forall\ \varepsilon>0\quad \exists\ A=A(t,\varepsilon)>0\quad$ s.t.

$$\sum_{|x|\leq A} p_t(x) \geqslant 1-\varepsilon .$$

...

$$\Psi_{N,t}([uN]) = \sum_{|x|\leq A} \rho_0(x/N)\, \underset{\displaystyle p_t^N(\underbrace{[uN]-x}_{:=z})}{\overset{\shortparallel}{p_t^N(x,[uN])}} + \sum_{|x|>A} \rho_0(x/N)\, p_t^N(x,[uN])$$

$$= \sum_{z\in B_A([uN])} \rho_0\left(\frac{[uN]}{N}-\frac{z}{N}\right) p_t^N(z) + \sum_{z\in B_A^c([uN])} \rho_0\left(\frac{x}{N}\right) p_t^N(x,[uN])$$

[ball of radius A around $[uN]$]

$$\left[\begin{aligned}\Psi_{N,t}([uN]) &= \sum_{x\in\mathbb{T}_N^d} \rho_0(x/N)\, p_t^N(\underbrace{[uN]-x}_{:=z}) \\ &= \sum_{z\in\mathbb{T}_N^d} \rho_0\left(\frac{[uN]}{N}-\frac{z}{N}\right) p_t^N(z)\end{aligned}\right]$$

$$\rightarrow \rho_0(u)(1-\varepsilon) \leq \lim_{N\to\infty} \Psi_{N,t}(x) \leq \rho_0(u)(1-\varepsilon) + \max_{x\in[0,1]} \rho_0(x)\,\varepsilon$$

ε can be made arbitrary small

$$\rightsquigarrow \lim_N \Psi_{N,t}([uN]) = \rho_0(u) \qquad \blacksquare$$

The macroscopic profile doesn't change:

"The dynamics does not have time to transport mass macroscopically in the time interval $[0,t]$".

From the point of view of RW:

Indeed because of tightness $\forall \varepsilon > 0$ and t fixed $\exists\, A = A(\varepsilon, t) > 0$ s.t.

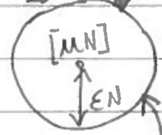
$$\mathbb{P}\left(|X_t| > A \right) \leq \varepsilon$$

this means that a profile at 0 stays at distance of order $1/N$ in $[0,t]$:

$$\frac{X_t}{N} \xrightarrow[N]{P} 0 .$$

"Physical Meaning":

$B^{\varepsilon N}([uN])$



ball around u macroscopically small but microscopically larger, i.e. of order εN with $\varepsilon \to 0$ and $N \to \infty$.

What I observe here at $\partial B^{\varepsilon N}([uN])$ in a time $[0,t]$ it happened inside $B^{\varepsilon N}([uN])$ close to $[uN]$ at a former time $\theta_N t$, where $\theta_N \xrightarrow[N]{} +\infty$.

θ_N is the time-scale:

⇝ We have also two time scale:

t microscopic time scale

$\theta_N t$ macroscopic time scale.

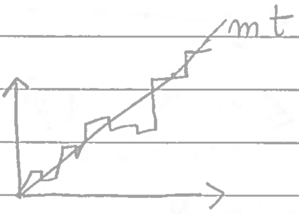
Define $m := \sum_{x \in \mathbb{Z}^d} x\, p(x)$

- $\neq 0$, asymmetric RWs
- $= 0$, $p(x) \neq p(-x)$ mean-zero asym. RWs
- $= 0$ $p(x) = p(-x)$ symmetric RWs

In next statement we consider t discrete time for simplicity.

$$p_t(x) = p^t(x) \leftarrow \boxed{\text{convolution of } t \text{ elementary transition probabilities of a particle}}$$

LLN for discrete asymmetric RWs
with time scale $\theta_N = N$:



$$1 = \lim_N P\left(\left| \frac{X_{tN}}{N} - mt \right| \leq \varepsilon \right) = \lim_N \sum_{x : \left|\frac{x}{N} - mt\right| \leq \varepsilon} p_{tN}(x)$$

For the density profile $\rho(u,t)$ of the particles system, we expect

$$\lim_N \Psi_{N, \underbrace{tN}_{\theta_N t \text{ hyperbolic time scale}}} ([uN]) = \rho_0(u - mt) := \rho(u,t), \text{ i.e.}$$

at time scale $\theta_N t = Nt$ we observe in u a new profile $\rho(u,t)$ that is the initial translated by mt (new local equilibrium) and $\rho(u,t)$ satisfies $\underbrace{\partial_t \rho + m \nabla \rho = 0}_{\text{PDE}}$.

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$$\underline{\text{Proof}}: \Psi_{N,tN}([uN]) = \sum_{x\in\mathbb{T}_N^d} \rho_0\left(\frac{x}{N}\right) p^N_{tN}\Big(\underbrace{[uN]-x}_{=:\,z}\Big)$$

$$= \sum_{z\in\mathbb{T}_N^d} \rho_0\left(\frac{[uN]-z}{N}\right) p^N_{tN}(z) =$$

$$= \sum_{z:\,\left|\frac{z}{N}-mt\right|\le\varepsilon} \rho_0\left(\frac{[uN]-z}{N}\right) p^N_t(z)$$

$$+$$

$$\sum_{z:\,\left|\frac{z}{N}-mt\right|>\varepsilon} \rho_0\left(\frac{[uN]-z}{N}\right) p^N_t(z)$$

$$\le \max_{x\in[0,1]} \rho_0(u) \sum_{z:\,|z/N-mt|>\varepsilon} p^N_t(z) \xrightarrow[N]{} 0$$

$y=-z+mtN$

$$= \sum_{\frac{|y|}{N}\le\varepsilon} \rho_0\left(\frac{[uN]-mtN}{N}+\frac{y}{N}\right) p^N_{tN}(-y+mtN)$$

$$= \rho_0\left(\frac{[uN]}{N}-mt\right) \underbrace{\sum_{\frac{|y|}{N}\le\varepsilon} p^N_{tN}(-y+mtN)}_{\xrightarrow[N]{} 1}$$

$$+ \sum_{\frac{|y|}{N}\le\varepsilon} \underbrace{O\left(\frac{y}{N}\right)}_{O(\varepsilon)} p^N_t(-y+mtN)$$

$$\Rightarrow \rho(u,t) = \lim_N \Psi_{N,Nt}([uN]) = \lim_N \rho_0\left(\frac{[uN]}{N}-mt\right)+O(\varepsilon)$$

$$= \rho_0(u-mt)$$

this can be made arbitrary small.

∎

If $m = 0 \rightsquigarrow \partial_t \rho = 0$, again it is necessary a larger time-scale to observe something, $\theta_N = N^2$.

diffusive time-scale.

$m = 0$ but

$$\sigma_{i,j} = \sum_{x \in \mathbb{Z}^d} x_i x_j \, p(x), \quad 1 \le i, j \le d$$

[cov. matrix not trivial $\neq 0$]

By CLT for discrete time RW [Donsker's theorem for continuous time case]

$$\frac{X_{tN}}{\sqrt{N}} \xrightarrow[N]{d} \mathcal{N}(0, t\sigma)$$

Hydrodynamics for density of particles

$$\rho(u,t) := \Psi_{N, N^2 t}([Nu]) = \sum_{x \in \mathbb{T}_N^d} \rho_0\left(\frac{x}{N}\right) P^N_{N^2 t}([Nu] - x)$$

$$= \sum_{z \in \mathbb{T}_N^d} \rho_0\left(\frac{[Nu] - z}{N}\right) P^N_{N^2 t}(z)$$

$$= E\left(\rho_0\left(\frac{[Nu] - X_{N^2 t}}{N}\right)\right)$$

$$\xrightarrow[\substack{\text{CLT} \\ \text{(Donsker)}}]{N} \int_{\mathbb{R}^d} \bar{\rho}_0(u-y) \, G^{\sigma t}(y) \, dy$$

gaussian distribution $\mathcal{N}(0, \sigma t)$

$$= \int_{\mathbb{R}^d} dx \, \rho_0(x) \, G^{\sigma t}(u-x) := \rho(u,t)$$

[Fundamental solution of heat equation]

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$$= \int_{\mathbb{R}^d} dx \; \bar{\rho}_0(x) \; G^{\sigma t}(u-x) \quad := \rho(u,t)$$

Fundamental sol. of heat equation

$\bar{\rho}_0 : \mathbb{R}^d \to \mathbb{R}$ is periodic function of period $\mathbb{T}^d$ and equal to ρ_0 on $\mathbb{T}^d$.

$\leadsto$ we have the following hydrodynamics:

$\rho(u,t)$ is sol. of

$$\begin{cases} \partial_t \rho = \sum_{1 \le i,j \le d} \sigma_{i,j} \, \partial_{u_i} \partial_{u_j} \rho \\ \rho(0,u) = \rho_0(u) \end{cases}$$

With the hydrodynamics at time scales $\theta_N = N, N^2$, we have showed the conservation of local equilibrium:

Given $(\mu_N)_{N \ge 1}$ initial distr. of loc. profile $\rho_0(\cdot)$

$= \nu^N_{\rho_0(\cdot)}$

s.t. we have the local equilibrium at $t=0$, i.e. $\lim_N \tau_{[uN]} \mu^N \overset{(w)}{=} \nu_{\rho_0(u)}$

$$\Rightarrow \lim_N S^N(t\theta_N)(\tau_{[uN]} \mu^N) = \nu_{\rho(t,u)}$$

evolution operator of the process $(\eta_t)_{t \ge 0}$

$\forall \; t \ge 0$ and u continuity point of $\rho(t,\cdot)$, with $\rho(t,\cdot)$ is sol. of a proper PDE.

LECTURE 2

Chap 2 KL: weak convergence + models

+

paper De Masi, Ferrari (1984).

Some topology

$\Sigma = \mathbb{N}^{\mathbb{Z}^d} \ni (\eta(x))_x$

$\mathbb{N}^{\mathbb{Z}^d}$ with product topology, it is metrizable:

$d(\cdot,\cdot)$ on $\mathbb{N}^{\mathbb{Z}^d}$

$$d(\eta,\xi) = \sum_{x\in\mathbb{Z}^d} \frac{1}{2^{|x|}} \frac{|\eta(x)-\xi(x)|}{1+|\eta(x)-\xi(x)|}$$

Σ is complete separable metric space

In this topology:

Prop. K compact subset of Σ

$\Longleftrightarrow$

K is closed and $\exists$ a collection of positive numbers $\{n_x, x\in\mathbb{Z}^d\}$ s.t. $\eta(x)\le n_x$ $\forall \eta\in K$ (boundedness)

$\mathcal{Y} := \left\{ \begin{array}{l} \text{space of cylinder function, i.e. functions} \\ \text{that depend on the configurations only through} \\ \text{the configurations on a finite set of coordinates} \end{array} \right\}$

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$\mathcal{M}_1(\mathbb{N}^{\mathbb{Z}^d}) := \left\{ \begin{array}{l} \text{space of probability measures on } \mathbb{N}^{\mathbb{Z}^d} \text{ with} \\ \text{weak topology} \end{array} \right\}$

$\boxed{\text{Def}}$ $\{\mu_K\}_K$ on $\mathbb{Z}^d$ $\xrightarrow[K]{W} \mu$ (weakly converges to μ)

if $E_{\mu_K}(f) \to E_\mu(f)$ $\forall f \in \mathcal{C}_b(\mathbb{N}^{\mathbb{Z}^d})$

$\boxed{\text{Prop.}}$ $\mu_K \xrightarrow[K]{W} \mu$ on $\mathbb{N}^{\mathbb{Z}^d} \iff E_{\mu_K}(\psi) \to E_\mu(\psi)$

$\forall$ bounded cylinder function ψ.

$\boxed{\text{Proof}}$ $\left[\begin{array}{l} \text{let's do it with } K = \{0,1,\dots,k\}^{\mathbb{Z}^d} \text{ and } \mathcal{M}_1(K) \\ \text{instead of } \mathbb{N}^{\mathbb{Z}^d} \text{ and } \mathcal{M}_1(\mathbb{N}^{\mathbb{Z}^d}) \end{array} \right]$

$(\Leftarrow)$

$\{0,\dots,k\}^{\mathbb{Z}^d}$ is a compact space by definitions.

Remind (Heine-Cantor theorem): $f : \Omega_1 \to \Omega_2$ metric spaces with Ω_1 compact and f continuos

$\Rightarrow f$ is uniformly continuous, i.e.

$\left[\forall \varepsilon > 0 \ \exists \delta > 0 \text{ s.t. } d(\xi, \eta) < \delta \to |f(\xi) - f(\eta)| < \varepsilon \right]$

take $\Lambda \subset \mathbb{Z}^d$ finite s.t. if $\xi(x) = \eta(x)$ $\forall x \in \Lambda$

$\Rightarrow d(\xi, \eta) < \delta$. Given $f \in \mathcal{C}_b$, on Λ I define ψ cylindrical

s.t. on $\underline{\eta} = \underbrace{\underline{\eta}_\Lambda}_{\text{configuration restricted to } \Lambda} \overbrace{\underline{\eta}^*}^{\text{fixed configuration}}$

$$\psi(\underline{\eta}_\Lambda \underline{\eta}^*) = f(\underline{\eta}_\Lambda \underline{\eta}^*) \quad \forall \underline{\eta}_\Lambda$$

$$\sup_{\xi \in K} |\psi(\xi) - f(\xi)| = \sup_{\xi \in K} |f(\underbrace{\eta}_{:= (\underline{\xi}_\Lambda, \underline{\eta}^*)}) - f(\xi)| \le \varepsilon$$

Then $\left|\int f\,d\mu_K - \int f\,d\mu\right| \leq \left|\int f\,d\mu_K - \int \psi\,d\mu_K + \int \psi\,d\mu_K - \int \psi\,d\mu\right.$

$\left.+ \int \psi\,d\mu - \int f\,d\mu\right| \leq \int |f-\psi|\,d\mu_K + \left|E_{\mu_K}(\psi) - E_\mu(\psi)\right|$

$+ \int |f-\psi|\,d\mu \leq 3\underset{\uparrow\text{small at pleasure}}{\varepsilon}$.

$\Rightarrow$) direct by def.

For $\mathbb{N}^{\mathbb{Z}^d}$ one has to use the fact "Billingsley" at page 5 back of Lecture 1 to have uniform tightness of the family $\{\mu_K\}_K$ on a compact K_ε and Heine-Cantor on K_ε.

Prop. $\mu_K \xrightarrow[K]{w} \mu$ on $\mathbb{N}^{\mathbb{Z}^d} \Leftrightarrow \forall\, \Lambda \subset \mathbb{Z}^d$ finite and every sequence $\{a_x, x\in\Lambda\}$

$\mu_K\{\eta,\ \eta(x) = a_x,\ x\in\Lambda\} \xrightarrow[K]{} \mu\{\eta,\ \eta(x) = a_x\ \ \forall x\in\Lambda\}$

Proof $\left[\begin{array}{l}\text{let's do it with } K=\{0,1,\dots,k\}^{\mathbb{Z}^d} \text{ and } \mathcal{M}_1(K) \\ \text{instead of } \mathbb{N}^{\mathbb{Z}^d} \text{ and } \mathcal{M}_1(\mathbb{N}^{\mathbb{Z}^d})\end{array}\right]$

$\Rightarrow$) We have $\mu_K(\psi) \to \mu(\psi)$, take

$\psi(\cdot) = 1_{\{\eta(x) = a_x,\ x\in\Lambda\}}(\cdot)$.

$\Leftarrow$) Now take ψ general cylinder function, $\mathcal{D}(\psi) = \Lambda < +\infty$.

$\psi(\cdot) = \sum_{\underline{a}_\Lambda} \psi(\underline{a}_\Lambda)\, 1_{\{\eta(x)=a_x,\ \forall x\in\Lambda\}}(\cdot)$,

$\oplus$ $\mu_K(\psi) = \sum_{\underline{a}_\Lambda} \psi(\underline{a}_\Lambda)\, \mu_K\{\eta(x) = a_x,\ \forall x\in\Lambda\}$

$= \sum_{\underline{a}_\Lambda} \psi(\underline{a}_\Lambda)\, \mu\{\eta(x) = a_x,\ \forall x\in\Lambda\}$

$= \mu(\psi)$ ∎

20 In the case $\mathbb{N}^{\mathbb{Z}^d}$ ⊕ can not be written directly with "convex" combinations of Ψ because the sum it is on infinite index.

This extends to $\mathbb{N}^{\mathbb{Z}^d}$ again using tightness of "Billingsley" pg. 5 back and Heine-Cantor.

Remark : We often consider measures μ_N on $\mathbb{N}^{\mathbb{T}_N^d}$ that converges with N becoming large to a measure on $\mathbb{N}^{\mathbb{Z}^d}$. Let's define this.

Given μ_N we extend it with $\tilde{\mu}_N$ on $\mathbb{N}^{\mathbb{Z}^d}$: [with product topology]

$$\tilde{\mu}_N\left\{\eta : \eta(x) = a_x,\ x \in \left[-N/2, N/2\right]^d\right\}$$

$$= \mu_N\left\{\eta : \eta(x) = a_x,\ x \in \mathbb{T}_N^d\right\} + \text{periodic measure}$$

i.e. $\tilde{\mu}_N\{\eta : \eta(x) = \eta(x+Ny),\ y \in \mathbb{Z}^d\} = 1$

$\tilde{\mu}_N$ is concentrated on periodic configuration.

So the question $\tilde{\mu}_N \xrightarrow{w} \mu$ is well posed:

$$E_{\tilde{\mu}_N}(\Psi) = E_{\mu_N}(\Psi) \xrightarrow[N]{w} E_\mu(\Psi)$$

N s.t. $\mathcal{D}(\Psi) \subset \mathbb{T}_N^d$

Now we introduce some interacting particles systems tipically studied in scaling limits. While in Lecture 1 the RWs were moving freely without carying about other particles, now a particle jumps on the lattice being influenced by other particles.

$\boxed{\text{Def.}}$ (Simple exclusion process)

$\Sigma_N = \{0,1\}^{\mathbb{T}_N^d}$ p finite range, translational invariant, irreducible probability on $\mathbb{Z}^d$, i.e.

$p(x,y) = p(0,y-x) := p(y-x)$

$\sum_z p(z) = 1$, $p(x) = 0$ for $x_i > A$ $\forall\, i \in \{1,\dots d\}$

$$(L_N f)(\eta) := \sum_{x\in\mathbb{T}_N^d} \sum_{z\in\mathbb{T}_N^d} \eta(x)\big(1-\eta(x+z)\big)\, p^N(z) \cdot$$

$$\left[f(\eta^{x,x+z}) - f(\eta) \right] \text{, where}$$

$$\eta^{x,y}(z) = \begin{cases} \eta(z) &, z \neq x,y \\ \eta(x)-1 &, z = x \\ \eta(x)+1 &, z = y \end{cases}, \quad p^N(z) := \sum_{y\in\mathbb{Z}^d} p(z+yN)$$

if $p(z) = -p(z)$ it is a symmetric simple exclusion.

Remarks: • for N large enough $p(\cdot) = p^N(\cdot)$

• It is a conservative dynamics, irreducible on the hyperplane $\Sigma_N^K = \{ \underline{\eta} \in \mathbb{T}_N^d \mid \sum_{x\in\mathbb{T}_N^d} \eta(x) = K \}$

• $\nu_\alpha^N(\eta) = \prod_{x\in\mathbb{T}_N^d} \alpha^{\eta(x)} (1-\alpha)^{(1-\eta(x))}$, Bernoulli prod. measure of parametre α is invariant for SEP and reversible if $p(z) = p(-z)$, analogy

ν_α^N Bern. for SEP $\qquad\qquad$ ν_α^N Poisson for IRWs

namely:

- We have a natural parametrization in terms of particles density

$$\langle n(x) \rangle_{\nu^N_{\rho(\cdot)}} = \rho(x) .$$

- <u>Equivalence of ensemble</u> :

canonical measure $\nu_\alpha\Big(\cdot \;\Big|\; \sum_{x \in \mathbb{T}^d_N} \eta(x) = \underset{[\beta N^d]}{\overset{\shortparallel}{K}} \Big)$

converges to $\xrightarrow[N]{W}$

grand canonical measure $\nu_\beta(\cdot)$

$\boxed{\text{Def.}}$ [<u>Zero Range Process</u>]

$\Sigma_N = \mathbb{N}^{\mathbb{T}_N^d}$, $g: \mathbb{N} \to \mathbb{R}_+$, $g(0)=0$, $g(\underset{\overset{\Downarrow}{}}{\overset{\mathbb{N}}{n}} \neq 0) > 0$

$p(\cdot,\cdot)$ Finite range, irreducible, translational invariant transition probability on $\mathbb{Z}^d$.

With bounded variation, i.e. $g^* := \sup_{k \geq 0} |g(k+1) - g(k)| < +\infty$

$$(L_N f)(\eta) = \sum_{x \in \mathbb{T}^d_N} \sum_{z \in \mathbb{T}^d_N} p^N(z)\, g(\eta(x)) \left[\left(f(\eta^{x,x+z}) - f(\eta) \right) \right],$$

$$Z(\varphi) := \sum_{k \geq 0} \frac{\varphi^k}{g(k)!} , \quad g(k)! = \prod_{1 \leq j \leq k} g(j) \text{ and } g(0)! = 1$$

with radius of convergence φ^* $\rightsquigarrow$ $Z(\varphi)$ is analytic and strictly increasing on $[0, \varphi^*)$.

[extra assumption : $\lim_{\varphi \uparrow \varphi^*} Z(\varphi) = \infty$]

$$\mu_\varphi^N(\eta) = \prod_x \mu_x^N(\eta) = \prod_x \frac{1}{Z(\varphi)} \frac{\varphi^{\eta(x)}}{g(\eta(x))!}$$

is invariant and reversible if $p(z) = p(-z)$

We want to work with invariant measures described in terms of the particle density ρ

$$\rho = R(\varphi) := E_{\mu_\varphi^N}(\eta(0)) = \frac{1}{Z(\varphi)} \sum_{k \geq 0} k \frac{\varphi^k}{g(k)!}$$

density (ρ), chemical potential (φ)

strictly increasing $\rightsquigarrow \exists$ inverse Φ s.t.

$\Phi(\rho) = \varphi$ and $\rho = E_{\mu^N_{\Phi(\rho)}}(\eta(0)) = E_{\mu_\rho^N}(\eta(0))$

$E_{\mu_\rho}(g(\eta(0)))$

$\boxed{\text{Prop.}}$ (Boundary driven zero-range)

$\Lambda_N = \mathbb{Z}^d \cap N\Omega, \quad \Omega \subset \mathbb{R}^d$

$$L_N f(\eta) = \frac{1}{2} \sum_{\substack{x,y \in \Lambda_N \\ |x-y|=1}} g(\eta(x))[f(\eta^{x,y}) - f(\eta)]$$

$$+ \frac{1}{2} \sum_{\substack{x \in \Lambda_N \\ y \notin \Lambda_N \\ |x-y|=1}} g(\eta(x))[f(\eta^{x,-}) - f(\eta)] + \psi\left(\frac{y}{N}\right)[f(\eta^{x,+}) - f(\eta)]$$

has invariant measure $\mu_N(\eta) = \prod_{x \in \Lambda_N} \frac{\phi_N(x)^{\eta(x)}}{Z(\phi_N(x))\, g(\eta(x))!}$

where $\eta^{x,\pm}(y) := \begin{cases} \eta(y) \ , & y \neq x \\ \eta(x) \pm 1 \ , & y = x \end{cases}$

Ψ regular function and $\phi_N(x)$ to be determined

[De Masi, Ferrari (1984)]

Proof

It holds $g(k) \dfrac{\varphi^k(x)}{g(k)!} \dfrac{\varphi^j(y)}{g(j)!} = g(j+1) \dfrac{\varphi^{k-1}(x)}{g(k-1)!} \dfrac{\varphi^{j+1}(y)}{g(j+1)!} \dfrac{\varphi(x)}{\varphi(y)}$

We want to verify

$$0 = \langle L_N f \rangle_{\mu_N} = \frac{1}{2} \sum_{\eta} \mu_N(\eta) \Bigg\{ \sum_{\substack{x,y \in \Lambda_N \\ |x-y|=1}} g(\eta(x)) \left(f(\eta^{x,y}) - f(\eta) \right)$$

$$+ \sum_{\substack{x \in \Lambda_N, \\ y \notin \Lambda_N \\ |x-y|=1}} g(\eta(x)) \left(f(\eta^{x,-}) - f(\eta) \right) + \Psi\left(\frac{y}{N}\right) \left(f(\eta^{x,+}) - f(\eta) \right) \Bigg\}$$

[Remind that $\eta^{x,y}(z) := \begin{cases} \eta(z) \ , & x,y \neq z \\ \eta(x)-1 \ , & z = x \\ \eta(y)+1 \ , & z = y \end{cases}$]

With the changes of variables $\eta' = \eta^{x,y}$, $\tilde{\eta} = \eta^{x,+}$ and $\hat{\eta} = \eta^{x,-}$ I have respectively

$$\mu_N(\eta) = \mu_N(\eta') \frac{\phi_N(x)}{\phi_N(y)} \frac{g(\eta'(y))}{g(\eta'(x)+1)}$$

$$\mu_N(\eta) = \mu_N(\tilde{\eta}) \frac{g(\tilde{\eta}(x))}{\phi_N(x)} \ , \quad \mu_N(\eta) = \mu_N(\hat{\eta}) \frac{\phi_N(x)}{g(\hat{\eta}(x)+1)}$$

$$\langle L_N f \rangle_{\mu_N} = \frac{1}{2} \sum_{\eta} \mu_N(\eta) f(\eta) \Big\{ \sum_{x \in \Lambda_N} g(\eta(x)) \sum_{y \in \Lambda_N : |x-y|=1} \left[\frac{\phi_N(y)}{\phi_N(x)} - 1 \right]$$

$$+ \sum_{x \in \Lambda_N} \sum_{y \notin \Lambda_N : |x-y|=1} \left[\phi_N(x) - \Psi(y/N) \right]$$

$$+ \sum_{x \in \Lambda_N} g(\eta(x)) \sum_{y \notin \Lambda_N : |x-y|=1} \left[\frac{\Psi(y/N)}{\phi_N(x)} - 1 \right] \Big\} = 0$$

We have stationarity if ϕ_N s.t.

$$\boxplus \begin{cases} \displaystyle\sum_{\cancel{y:|y-x|=1}} [\phi_N(y) - \phi_N(x)] = 0 \quad , \forall x \in \Lambda_N . \\ \phi_N(x) = \Psi\left(\frac{x}{N}\right) , \forall x \notin \Lambda_N, \ \cancel{\exists\, y \in \Lambda_N \text{ s.t. } |x-y|=1} . \end{cases}$$

$\Rightarrow \ = \frac{1}{2} \Delta_N \phi_N(x) = 0$ ~~we have a discrete~~ $\rightsquigarrow$ discrete Dirichlet problem.

where $\Delta_N \phi_N(x) := \phi_N\left(x + \frac{1}{N}\right) + \phi\left(x - \frac{1}{N}\right) - 2\phi(x)$

$$\Rightarrow \sum_{x \in \Lambda_N} \Delta_N \phi(x) = \sum_{x \in \Lambda_N, y \notin \Lambda_N} \phi_N(y) - \phi_N(x) = 0$$

[ϕ_N is an harmonic function]

$\rightsquigarrow$ We have stationarity if ϕ_N solve the Dirichelet problem $\boxplus$.

■

<u>Remark</u>

The process is <u>reversible</u> $\Leftrightarrow \Psi\left(\frac{y}{N}\right) = \Psi$ constant $\forall y \in \partial \Lambda_N$. In this case ϕ_N is ~~the constant~~ function Ψ on Λ_N ~~because ϕ_N is harmonic.~~

LECTURE 3

Chap. 3 KL: Weak formulation of local equilibrium

Def. (Product measures with slowly varying parameter associated to a profile ρ)

Given a smooth profile $\rho : \mathbb{T}^d \longrightarrow \mathbb{R}_+$
$\nu^N_{\rho(\cdot)}$ is product measure on Σ_N [state space] marginals
$\Sigma_N = \Sigma^{\mathbb{T}^d_N}$

$$\nu^N_{\rho(\cdot)}\{\eta,\ \eta(x)=k\} = \nu^N_{\rho(\frac{x}{N})}\{\eta,\ \eta(0)=k\},\ \forall x\in\mathbb{T}^d_N,\ \forall k\geq 0$$

with $\nu^N_{\rho(\cdot)}(\eta) = \prod_{x\in\mathbb{T}^d_N} \nu^N_{\rho(\frac{x}{N})}(\eta)$

Example: Bernoulli, i.e.
(on $\Sigma_N = \{0,1\}^{\mathbb{T}^d_N}$)

$$\nu^N_{\rho(\frac{x}{N})}(\eta) = \prod_{x\in\mathbb{T}^d_N} \rho\left(\frac{x}{N}\right)^{\eta_0(x)} \left(1-\rho_0\left(\frac{x}{N}\right)\right)^{(1-\eta_0(x))}$$

Remark: Tipically ρ is the density profile characterizing η, i.e. $\langle \eta(x)\rangle = \rho\left(\frac{x}{N}\right)$

In this case by Chebychev

$$\lim_{N} \nu^N_{\rho(\cdot)}\left[\left|\frac{1}{N^d}\sum_{x\in\mathbb{T}^d_N} G\left(\frac{x}{N}\right)(\tau_x\psi)(\eta) - \int_{\mathbb{T}^d} du\, G(u)\,\tilde{\Psi}(\rho(u))\right| > \delta\right] = 0 \quad (*)$$

$\forall\, G : \mathbb{T}^d \to \mathbb{R}$ continuous, ψ bounded cylinder function and $\forall\, \delta > 0$, where

$$\tilde{\psi}(\rho) = E_{\nu_\rho}[\psi] = \int_{\Sigma_N} \psi(\eta)\, \nu_\rho(d\eta).$$

$\uparrow$ constant

Def. Given an initial measure $(\mu_N)_{N\geq 1}$ on Σ_N if it is true ⊛ for $(\mu_N)_{N\geq 1}$ we say that we have a <u>weak</u> <u>local equilibrium</u>.

In chapter 1 we had the motion of <u>strong</u> local equilibrium, namely

$$\lim_N \tau_{[uN]} \mu^N \overset{(w)}{=} \nu_{\rho_0(u)} \leftarrow \left[\begin{array}{l}\text{equilibrium} \\ \text{measure of} \\ \text{constant profile} \\ \rho_0(u)\end{array}\right]$$

for some μ^N initial measure.

At time $t > 0$ we still had local equilibrium if

$$\lim_N \underset{\substack{\uparrow \\ \text{evolution} \\ \text{operator of } (\eta_t)_{t\geq 0}}}{S^N}(t\,\underset{\substack{\uparrow \\ \text{time scaling}}}{\theta_N})\left(\tau_{[uN]}\, \mu^N\right) = \nu_{\rho(t,u)}$$

where $\rho(t,u)$ was solution of a PDE.

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Now we have <u>conservation of local equilibrium</u> if at time $t>0$

$$\lim_N \mathbb{P}_{\mu_N}\left[\left|\frac{1}{N^d}\sum_{x\in\mathbb{T}^d_N} G\left(\frac{x}{N}\right)(\tau_x\Psi)(\eta_{t\theta_N}) - \int_{\mathbb{T}^d} du\, G(u)\Psi(\rho(u,t))\right| > \delta\right] = 0$$

[~~induced measure~~

$\mu^N \circ \pi_N^{-1}(\eta_t, du)$

where $\pi_N(\eta_t, du) := \frac{1}{N}\sum_{x\in\mathbb{T}^d_N} \eta_t(x)\,\delta_x(du)$ is the <u>empirical measure</u>.]

and $\rho(t,u)$ is the solution of a PDE.

<u>Remark</u> : taking $\Psi(\eta) = \eta(0)$ we have the statement of hydrodynamics.

$\boxed{\text{Prop.}}$ Let $(\mu^N)_{N\geq 1}$ a local equilibrium (strong) of profile ρ a.s. continuous on $\mathbb{T}^d$ $\Rightarrow$ μ^N is a <u>weak</u> local equilibrium of profile ρ.

Proof: the proof comes from the fact that spatial averages over macroscopically small but microscopically large boxes around a point $[uN]$ can be replaced (in probability) by average respect on equilibrium measure $\nu_{\rho(u)}$ of constant profile $\rho(u)$

Given $[Nu]_1 < x < [Nu] + \frac{1}{N}$

$$E_{\mu N}\left[\left|\frac{1}{(2l+1)^d}\sum_{|y-x|\leq l}\tau_y\psi(\eta) - \tilde{\psi}\left(\rho\left(\frac{x}{N}\right)\right)\right|\right]$$

by local equilibrium

$$\xrightarrow[N]{\Downarrow} = E_{\nu_{\rho(u)}}\left[\left|\frac{1}{(2l+1)^d}\sum_{|y|\leq l}\tau_y\psi(\eta) - \tilde{\psi}(\rho(u))\right|\right]$$

$$\xrightarrow[l\to\infty]{\text{by LLN}} 0.$$

$\blacksquare$

(30)

CURRENTS

[from "Microscopic and Macroscopic perspectives on non-eq stationary states" Leonardo De Carlo "ArXiv"
+
"Large Scale Dynamics of Interacting Particles" Spohn Section 2.2-2.3 in Part II]

Keep in mind models with discrete particles, like exclusion process, zero range, etc.. Consider jump on nearest neighbours.

Def. (istantaneous current)

$$j_\eta(x,y) := \underbrace{c_{x,y}(\eta)}_{} - c_{y,x}(\eta) \quad , x,y \in \mathbb{T}^d_N.$$

rate of jumps between from x to y and from y to x.

Physical meaning: rate at which particles cross the bond (x,y).

Def. (current or current bond)

$$J_t(x,y) := \underbrace{N_t(x,y)}_{\substack{\#\text{ of particles}\\ \text{that crossed }(x,y)\\ \text{in }[0,t]}} - \underbrace{N_t(y,x)}_{\substack{\#\text{ of particles}\\ \text{that crossed }(y,x)\\ \text{in }[0,t]}}$$

Remarks • $j_{\eta_t}(x,y)$ is a function of the configuration at time t.

• $J_t(x,y)$ is a function of the trajectory $\{\eta_s\}_{0\le s\le t}$.

• $j_\eta(x,y)$ and $J_t(x,y)$ are discrete vector field:

$$j_\eta(x,y) = -j_\eta(y,x) \quad , \quad J_t(x,y) = -J_t(y,x)$$

For the joint Markov process $\tilde{\eta}(t) = \{\eta(t), N(t)\}$,

$\tilde{\eta} \in \Sigma_N \times \mathbb{Z}^{E_N}$, where $E_N := \{\text{set of all bonds } (x,y),\ x,y \in \mathbb{T}_N^d\}$,

$N = \{N_{x,y}\}_{(x,y)\in E_N}$ defined by the Markov generator $\tilde{L}_N$

$$\tilde{L}_N F(\eta, N) = \sum_{(x,y)\in E_N} c_{x,y}(\eta)\left[F(\eta^{x,y}, N+\delta_{xy}) - F(\eta, N)\right]$$

$$(N+\delta_{x,y})_{z,w} := \begin{cases} N_{z,w} & , z,w \neq x,y \\ N_{z,w}+1, & z,w = x,y \end{cases}$$

take $F(\eta, N) = N(x,y) - N(y,x)$

$$\rightsquigarrow \tilde{L}_N F(\eta, N) = c_{x,y}(\eta)(+1) + c_{y,x}(-1)$$
$$= c_{x,y}(\eta) - c_{y,x}(\eta)$$
$$= j_\eta(x,y)$$

Choosing

$F(N_t) = J_t(x,y) \rightsquigarrow F(N_0) = 0$ and

$$\Rightarrow M_t(x,y) \underset{\triangle}{=} J_t(x,y) - \int_0^t ds\ j_{\eta_s}(x,y)$$

is a Martingale (from Dynkin's formula)

From this formula it follows a more general definition of $j_\eta(x,y)$ [generalizable to other kind of particles systems, e.g. for not discrete particles].

Because $\triangle$ allows to treat the difference between $J_t(x,y)$ and $\int_0^t ds\ j_{\eta_s}(x,y)$ as a microscopic fluctuation giving the definition:

$$j_\eta(x,y) := \lim_{t \to 0} \frac{\mathbb{E}^\eta(J_t(x,y))}{t}$$

expectation of $\mu_N \circ \eta_t^{-1}$ with $\mu_N = \delta_\eta$

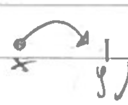
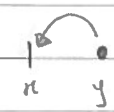
$$\mathbb{E}^\eta(J_t(x,y)) = \int \mathbb{P}^\eta(d\{\eta_t\}_t)\ J_t(x,y)$$

integration over all trajectories from η at time 0.

to compute $j_\eta(x,y) = c_{x,y}(\eta) - c_{y,x}(\eta)$ use the fact that the probability to have two jumps in $[0,t]$ is $O(t^2)$

and the fact that $\lim_{t \to 0} \frac{\mathbb{P}^\eta(\eta_t = \eta')}{t} =$

$= \lim_t \frac{p_t(\eta,\eta')}{t} = c(\eta,\eta')$ and

$J_t(x,y) = +1$ if  -1 if 

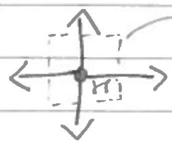
and 0 otherwise.

Microscopic mass conservation law :

$\phi : E_N \to \mathbb{R}$ is a discrete vector field

if $\phi(x,y) = -\phi(y,x)$,

$$(\mathrm{div}\, \phi)(x) := \sum_{y : (x,y) \in E_N} \phi(x,y)$$



discrete divergence computed in this box.

We have the following microscopic version of $\partial_t \rho + \mathrm{div}(J(\rho)) = 0$:

$$\boxed{\begin{aligned} &\eta_t(x) - \eta_0(x) + \mathrm{div}\, J_t(x) = 0 \\ \rightsquigarrow\ &\eta_t(x) - \eta_0(x) + N^2 \int_0^t ds\, (\mathrm{div}\, j_{\eta_s})(x) + \mathrm{div}\, M_t(x) = 0 \end{aligned}}$$

We will see that this allows to know

the euristhic hydrodynamics giving an equivalent of Dynkin's formula, but without the needs of the computations of the application of generator to configurations.

LECTURE 4

Chapter 4 (KL)

Hydrodynamics Equation of Symmetric Simple Exclusion Processes (SSEP)

We study the H.D. of the process on $\mathbb{T}_N^d$ defined by the Markov Generator

$$\mathcal{L}_N f(\eta) := \sum_{(x,y)\in E_N} \eta(x)(1-\eta(y)) [f(\eta^{x,x+1}) - f(\eta)]$$

and state space $\Sigma_N = \{0,1\}^{\mathbb{T}_N^d}$.

[Def.] (empirical measures) [space of positive measures on $\mathbb{T}^d$]

$$\pi_t^N(du, \eta) := \frac{1}{N} \sum_{x\in\mathbb{T}_N^d} \eta_t(x)\, \delta_{x/N}(du) \in \mathcal{M}_+(\mathbb{T}^d)$$

We call $\mathcal{D}([0,T], \Sigma_N)$ the space of trajectories of the process $\{\eta_t\}_t$ and with $\mathcal{D}^+([0,T], \mathcal{M}_+(\mathbb{T}^d))$ of CADLAG trajectories from $[0,T]$ to $\mathcal{M}_+(\mathbb{T}^d)$.

[Def] Let $\{\mathbb{P}_N\}_{N\geq 0}$ the sequence of probability measure on $\mathcal{D}([0,T], \Sigma_N)$ induced by the process $\eta_{N^2 t}$ with initial distribution μ_N.

diffusive time-scale

Remark We will need $\theta_N = N^2$ as time-scale to observe a non-trivial H.D. (the process is symmetric w.r.t. to bernoulli measure). Equivalently we can consider the accelerated process $N^2 \mathcal{L}_N$.

Def. Let Q_N the probability measure on $D^+([0,T], \mathcal{M}_+(\mathbb{T}^d))$ induced by the Markov process $\pi^N_t(du, \eta)$, i.e.
$$Q_N := \mu_N \circ (\pi^N_t)^{-1}.$$

Statement of H. D. for SSEP

Let $\rho_0 : \mathbb{T}^d \to [0,1]$ be an initial density profile and let μ^N be the sequence of Bernoulli product measures of slowing varying parameter associated to the profile ρ, i.e. $\mu^N\{\eta ; \eta(x) = 1\} = \rho_0(\frac{x}{N})$, ⊞

then $\forall\, t > 0$, the sequence of random measure $\pi^N_t(du, \eta)$ converges in Probability to the absolute continuous measure $\rho(t,u)\,du$, namely
$$\lim_{N\to\infty} Q_N\left(\left| \int_{\mathbb{T}^d} f \, d\pi_N(\eta_t) - \int_{\mathbb{T}^d} f \, \rho(u,t)\,du \right| > \varepsilon \right) \overset{(\Delta)}{=} 0, \ \forall \varepsilon > 0$$

where $\rho(u,t)$ is the solution (strong) of the Cauchy problem
$$\begin{cases} \partial_t \rho = \Delta \rho_t ; \\ \rho(0) = \rho_0 \end{cases}$$

Remarks: 1) ⊞ implies ⊗ (Chebychev) at time $t=0$ with $\rho(u,0)=\rho_0(u)$.

2) ⊞ at time $t=0$ is what at page 26 is called (weak) local equilibrium and for $t>0$ it is the conservation of weak local equilibrium.

3) At page 29 it is discussed that the strong local equilibrium of $(\mu_N)_{N\geq 1}$

[$\lim_N S^N(t\theta_N)([\tau_{[uN]}\mu^N]) = \nu_{\rho(t,u)}$, $\forall\ t\geq 0$

we proved it in lecture 1 for IRWs]

implies the weak local equilibrium of μ_N.

The proof it is based on the limit of page 29 where it is showed that spatial [averages over macroscopically small but microscopically large boxes can be replaced (in probability) by average respect an equilibrium measure $\nu_{\rho(u)}$ of constant profile $\rho(u)$.

This is the physical meaning of weak local equilibrium and of the proof of Replacement lemmas.

Steps of the Proof :

1) Proving that $\{Q_N\}_{N\geq 0}$ is relatively compact (Prohorov's theorem)

2) Proving that all converging subsequences $\{Q_{N_k}\}_{k\geq 0}$ have the same limit.

Remarks : A) 1) requires to set a proper topology $D([0,T], \mathcal{M}_+)$ to characterize compact sets.

B) 2) will be given by the typical behaviour [in probability LLN] of $\pi_t^N(\eta)$ for N large, i.e. the PDE of the H.D.

$\boxed{\text{Topology and Compactness}}$ [Billingsley, "Convergence of Probability measure" Chap. 3 Sect. 12]

$\mathcal{M}_+(\mathbb{T}^d)$ with weak topology is a separable complete metric space with the metric

$$\rho(\mu,\nu) := \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{|\langle \mu, f_k\rangle - \langle \nu, f_k\rangle|}{1+|\langle \mu, f_k\rangle - \langle \nu, f_k\rangle|}$$

$\{f_k\}_{k\geq 1}$ dense countable set of $C(\mathbb{T}^d)$ functions.

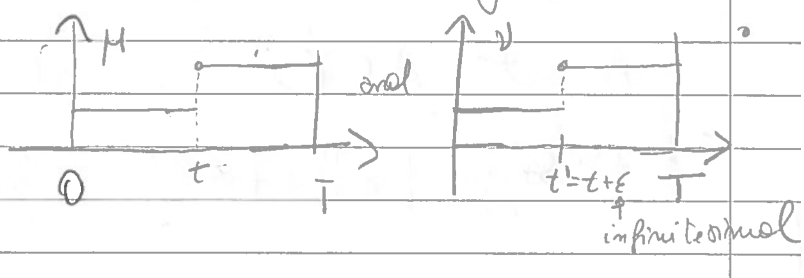
where $\langle f, \mu\rangle := \int_{\mathbb{T}^d} f \, d\mu$.

<u>Remark</u> : A family of measures $\mathcal{F} \subset \mathcal{M}_+(\mathbb{T}^d)$ is relatively compact if and only if

$$\sup_{\mu \in \mathcal{F}} \int_{\mathbb{T}^d} d\mu(x) < +\infty,$$

i.e. <u>total mass is finite</u>. $\rightsquigarrow$ for π_N if $\langle \pi_N, f \rangle < +\infty$ $\forall \eta$ $\Rightarrow$ the property it is deterministic, [e.g. like in exclusion processes].

The <u>Uniform topology</u> $d(\mu,\nu) = \sup_{s,t\in[0,T]} g(\mu(s),\nu(t))$ is not good because it evaluates as "very different" measures that differs for a small time difference in a jump, like



So it is introduced the <u>Skorohod Topology</u>, where two trajectories $\mu(s)$ and $\nu(s)$ are "close" if deforming a little bit at the same time their "ordinates/graph" and "time scale" they are "similar".

$$d(\mu,\nu) := \inf_{\lambda \in \Lambda} \max \left\{ \sup_{0\le t\le T} |\lambda t - t| , \sup_{0 \le t \le T} g\big(\mu(t), \nu(\lambda(t))\big) \right\}$$

where $\Lambda := \{$ strictly increasing functions $\lambda : [0,T] \to [0,T]$ with $\lambda(0) = 0$ and $\lambda(T) = T$ $\}$

$\boxed{\text{Prop.}}$ $\mathcal{D}^{+}([0,T], \mathcal{M}_{+})$ with the metric d is a separable metric space, to make it complete $\sup_{0\le t\le T} |\lambda t - t|$ in the previous definition has to be replaced by $\|\lambda\|^{\circ} = \sup_{s<t} \left| \log \frac{\lambda t - \lambda s}{t-s} \right|$

Remark : in this norm λ is close to the identity if the slope $\frac{\lambda t - \lambda s}{t-s} \simeq 1$, i.e. if the logarithm of the slope is close to 0.

$\boxed{\text{Th.}}$ (Ascoli-Arzelà) Consider the space of continuous function $C = C[0,1]$ with the uniform Topology, i.e. $\rho(x,y) = \|x-y\| = \sup_t |x(t) - y(t)|$

A set $A \subset C$ is relatively compact $\iff$

1) $\sup_{x\in A} \|x\| < +\infty$ (uniform boundedness property)

2) $\lim_{\delta\to 0} \sup_{x\in A} w_x(\delta) = 0$ (uniform equicontinuity property)

where $w_x(\delta) := \sup_{|t-s|\le\delta} |x(s) - x(t)|$, with $0<\delta<1$, is the modulus of continuity of $x(\cdot)$.

Remind : x is uniformly continuous iff over $[0,1]$ is $\lim_{\delta\to 0} w_x(\delta) = 0$.

Let's call $w_\mu(\delta) := \sup_{|t-s| \le \delta} g(\mu_s, \mu_t)$ modulus of uniform continuity.

$\boxed{\text{Prop.}}$ Given $A \subset \mathcal{D}^+([0,T], \mathcal{M}_+)$ if

1) $\{\mu_t ;\ \mu \in A,\ t \in [0,T]\}$ is relatively compact on $\mathcal{M}_+$ [<u>boundedness</u> property].

2) $\lim_{\delta \to 0} \sup_{\mu \in A} w_\mu(\delta) = 0$ [<u>uniform equicontinuity</u> property]

$\Rightarrow$ A is relatively compact

<u>Remarks</u> 1) $\{\mu_t\}_+ \in \mathcal{C}([0,T], \mathcal{M}_+)$ iff $\lim_{\delta \to 0} w_\mu(\delta) = 0$. So the previous propositions becomes $\Leftrightarrow$ if $A \in \mathcal{C}([0,T], \mathcal{M}^+)$.

This is the case of the typical case and the one of the book KL.

2) For $A \subset \mathcal{D}^+([0,T], \mathcal{M}_+)$ I have iff for the modified modulus of uniform continuity:

$$\tilde{w}_\mu(\delta) := \inf_{\{t_i\}_{0 \le i \le r}} \max_{0 \le i \le r} \sup_{t_i \le s < t < t_{i+1}} g(\mu_s, \mu_t)$$

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where the inf is over all partitions $\{t_i, 0 \leq i \leq r\}$ of the interval $[0,T]$ of size δ, i.e.

$$\begin{cases} 0 = t_0 < t_1 < \dots < t_r = 1 \\ t_i - t_{i-1} > \delta \quad , \quad i = 1, \dots, r \quad , \end{cases}$$

indeed $\mu \in \mathcal{D}^+([0,T], M_+) \Leftrightarrow \lim_{\delta \to 0} w_\mu(\delta) = 0.$

[Proof: comes from the fact the $\{\mu_t\}_t$ for $t \in [0,T]$ can have at most finitely many points t at which the jump $|\mu(t) - \lim_{s \uparrow t} \mu(s)|$ exceeds a given positive number]

3) $\tilde{w}_\mu(\delta) \leq w_\mu(2\delta)$, Proof:

For $\delta < T/2$ the interval $[0,T)$ can be split in subintervals satisfying $\delta < t_i - t_{i-1} \leq 2\delta$, $\rightsquigarrow$ we have $\tilde{w}_\mu(\delta) \leq w_\mu(\delta)$ if $\delta < T/2$.

$\boxed{\text{Prohorov's Theorem}}$

Let X complete separable metric space and $\{\mathbb{P}_N\}_N$ sequence of probability on it.

$\{\mathbb{P}_N\}_N$ is tight $\Leftrightarrow$ $\{\mathbb{P}_N\}_N$ is relatively compact

[$\exists$ $\{\mathbb{P}_{N_K}\}_{N_K}$ s.t. $\mathbb{P}_{N_K} \xrightarrow[K]{(w)} \mathbb{P}$, i.e.

$\lim_K \int_X f \, d\mathbb{P}_{N_K} = \int_X f \, d\mathbb{P}$

$\forall f \in C_b(X)$.]

Th. (Functional Prohorov)

Let $\{\mathbb{Q}_N\}_N$ a sequence of probability measures on $\mathcal{D}^+([0,T]; \mathcal{M}^+)$ if

1) $\forall\, t \in [0,T]$ and $\forall\, \varepsilon > 0$, there is a compact $K(t,\varepsilon) \subset \mathcal{M}_+$ s.t. $\sup_N \mathbb{Q}_N[\mu_t \notin K(t,\varepsilon)] \leq \varepsilon$,

i.e. we have tightness of $\{\mathbb{Q}_N\}_N$. [trajectory at time t]

2) $\forall\, \varepsilon > 0$, $\lim_{\delta \to 0} \limsup_{N \to \infty} \mathbb{Q}_N[\mu : w_\mu(\delta) > \varepsilon] = 0$,

i.e. we have equicontinuity property in probability, [all the trajectory $\{\mu_t\}_{t \geq 0}$]

$\Rightarrow$ $\{\mathbb{Q}_N\}_N$ is relatively compact. To have

$\Leftarrow$ I have to replace $w_\mu(\delta)$ with $\tilde{w}_\mu(\delta)$.

Meaning of what we have to do:

A) To have 1): $\lim_{\ell \uparrow \infty} \limsup_{N \to \infty} \mathbb{Q}_N \left\{ \sup_{0 \leq t \leq T} \frac{1}{N} \sum_{x=1}^{N} |\eta_t(x)| \geq \ell \right\} = 0$

If $\eta(x) \in \Sigma$, with $\Sigma \subset \mathbb{R}$ bounded $\rightsquigarrow$ A) it is deterministic.

B) To have 2): $\forall\, f \in C(\mathbb{T}^d)$ test function [modulus of uniform continuity $w_{\langle f, \pi^N \rangle}(\delta)$]

$$\lim_{\delta \downarrow 0} \limsup_{N \to \infty} \mathbb{Q}_N \left\{ \sup_{\substack{0 \leq t, s \leq T \\ |t-s| \leq \delta}} \left| \langle f, \pi_t^N \rangle - \langle f, \pi_s^N \rangle \right| > \varepsilon \right\} = 0$$

$\forall\, \varepsilon > 0$.

Therefore B) tells us that to verify 2) it is enough to have 2) for each projected process $\langle \pi^N, f \rangle$.

→ **Prop.** It is enough to prove B) for $\{f_k\}_k$ countable dense set of $C(\mathbb{T}^d)$.

$\mathbb{Q}_N$ has 2) if $\forall k$

$$\lim_{\delta \to 0} \limsup_{N \to \infty} \mathbb{Q}^k_N \left\{ W_{\langle \pi^N, f_k \rangle}(\delta) > \varepsilon \right\} = 0, \quad \forall \varepsilon.$$

[prob. induced by the process $\langle \pi^N, f_k \rangle$]

Proof:

Fix $\varepsilon > 0$, $\gamma > 0$ and take k_ε s.t. $\frac{\varepsilon}{2} \geq \frac{1}{2^{k_\varepsilon}}$.

$$\rightsquigarrow W_\mu(\delta) \leq \sum_{k=1}^{k_\varepsilon} \frac{1}{2^k} W_{\langle \mu, f_k \rangle}(\delta) + \sum_{k > k_\varepsilon} \frac{1}{2^k} =$$

$$= \sum_{k=1}^{k_\varepsilon} \frac{1}{2^k} W_{\langle \mu, f_k \rangle}(\delta) + \frac{1}{2^{k_\varepsilon}} \sum_{j=1} \frac{1}{2^j}$$

$$\leq \sum_{k=1}^{k_\varepsilon} \frac{1}{2^k} W_{\langle \mu, f_k \rangle}(\delta) + \frac{\varepsilon}{2}, \quad \text{Moreover}$$

$\exists\, \delta_0, N_0 \quad \forall\, N \geq N_0,\ k \leq k_\varepsilon,\ \delta < \delta_0$

$$\mathbb{Q}_N \left[W_{\langle \mu, f_k \rangle}(\delta) > \frac{\varepsilon}{2} \right] \leq \frac{\gamma}{2^k}.$$

Then $\mathbb{Q}_N \left[\sum_{k=1}^{k_\varepsilon} \frac{1}{2^k} W_{\langle \mu, f_k \rangle}(\delta) > \frac{\varepsilon}{2} \right]$

$$\leq \sum_{k=1}^{k_\varepsilon} \mathbb{Q}_N \left[\frac{1}{2^k} W_{\langle \mu, f_k \rangle}(\delta) > \frac{\varepsilon}{2} \right]$$

$$\leq \sum_{k=1}^{K_\varepsilon} \mathbb{Q}_N \left[w_{\langle \mu, f_k \rangle}(\delta) > \frac{\varepsilon}{2} \right]$$

$$\leq \sum_{k=1}^{K_\varepsilon} \frac{\gamma}{2^k} < \gamma \cdot \sum_{k=1}^{\infty} \frac{1}{2^k} = \gamma$$

From ⑥ $\mathbb{Q}_N \left[w_\mu(\delta) > \varepsilon \right] \leq$

$$\leq \mathbb{Q}_N \left(\sum_{k=1}^{K_\varepsilon} \frac{1}{2^k} w_{\langle \mu, f_k \rangle}(\delta) + \frac{\varepsilon}{2} \geq \varepsilon \right)$$

$$\leq \mathbb{Q} \left(\sum_{k=1}^{K_\varepsilon} \frac{1}{2^k} w_{\langle \mu, f_k \rangle}(\delta) \geq \frac{\varepsilon}{2} \right) < \gamma$$

■

$\boxed{\text{Proof}}$ of Hydrodynamics Statement at page 36

- Relative compactness : we show propo. at pag. 44.

Consider $C^2(\mathbb{T}^d)$ as dense set $C(\mathbb{T}^d)$.

$f \in C^2(\mathbb{T}^d)$, consider

$$\mathbb{Q}_N \left(\sup_{\substack{0 \leq t, s \leq T \\ |t-s| < \delta}} \left| \langle \pi^N_t, G \rangle - \langle \pi^N_s, G \rangle \right| > \varepsilon \right) \quad \textcircled{+}$$

From the martingale

$$M(t,s) = \langle \pi^N_t, f \rangle - \langle \pi^N_s, f \rangle - N^2 \int_s^t dr \, \mathcal{L}_N \langle \pi^N_r(\eta), f \rangle$$

We have

$$(4) \le Q_N\Big(\sup_{\substack{0\le t,s\le T\\ |t-s|<\delta}} \Big| M(t,s)\Big| + N^2\Big|\int_s^t \underbrace{L_N \langle \pi_r^N(\eta), f\rangle}\Big| \ge \varepsilon\Big)$$

$$= \frac{1}{2}\int_s^t dr \langle \pi_r^N, \Delta_N G\rangle$$

discrete Laplacian.

We have the following estimates 1) and 2):

1) $$\Big|\frac{1}{2}\int_s^t dr \; \langle \pi_r^N, \Delta_N G\rangle\Big| \le c(G)\,|t-s|$$

constant depending on G.

Remind: Given $M(t) = f(\eta_t) - f(\eta_0) - \int_0^t dr\, \mathcal{L}f(\eta_r)$

for a Markov process $\{\eta_t\}_{t\ge 0}$

$$N(t) \underset{(\cdot\cdot)}{=} M^2(t) - \int_0^t dr\, \underbrace{\left(\mathcal{L}f^2(\eta_r) - 2f(\eta_r)\mathcal{L}f(\eta_r)\right)}_{:= B(s)}$$

is also a martingale.

We have also

$$\left(Lf - 2fLf\right)(\eta) \overset{\triangle}{=} \sum_{\eta'} c(\eta,\eta')\left(f_{\eta'} - f_\eta\right)^2$$

2) Take $f(\eta_r) = \langle \pi_r^N(\eta), f\rangle$

$$\rightsquigarrow \mathbb{E}_{Q_N}\left(M^2(s,t)\right) = \mathbb{E}_{Q_N}\left(\int_s^t B_s\, ds\right) \le c(G)\,\frac{|t-s|}{N^d}$$

from $(\cdot\cdot)$ constant depending on G

from $\triangle$

From estimates 1) and 2) and

From Chebychev, monotonicity of p-norm in probability spaces

~~[i.e. $\|f\|_p = \int_\Omega |f|^p d\mu \leq \int_\Omega |f|^q d\mu$ for $1 \leq p < q$]~~

and Doob's inequality we have prop. at 44 of tightness when I take the limits.

- Limit of the converging subsequences:

$$\sup_{0 \leq t \leq T} |\langle \pi^N_t, G \rangle| \leq \frac{1}{N^d} \sum_{x \in \mathbb{T}^d_N} G(x/N) \underset{N \gg 1}{\simeq} \int_{\mathbb{T}^d} G(u) du + O\left(\frac{1}{N}\right)$$

$\rightsquigarrow$ denoting Q^* a limit point of a converging subsequence $\{Q_{N_k}\}_k$ $\quad Q_{N_k} \xrightarrow[k]{(w)} Q^*$

we have that

$$Q^* [\pi : \pi_t(du) = \pi_t(u) du] = 1 \quad , \text{ i.e.}$$

Q^* is concentrated on absolutely continuous trajectories w.r.t. to Lebesgue measure.

Now consider the martingale

$$\tilde{M}(t) = \langle \pi^N_t, G \rangle - \langle \pi^N_0, G \rangle - \int_0^t ds \left[\langle \pi^N_s, \partial_s G \rangle + N^2 \mathcal{L}_N \langle \pi^N_s, G \rangle \right]$$

defining martingales $\tilde{N}(t)$ and $\tilde{B}(s)$ analogous to $N(t)$ and $B(t)$ of previous page 46

we have $\limsup_{N \to \infty} Q^N (|\tilde{M}(t)| > \epsilon) = 0$

with analogous computations.

since $N^2 \mathcal{L}_N \langle \pi_s^N, G \rangle = \langle \frac{1}{2} \Delta G_N, \pi_s^N \rangle$

the RHS of $\tilde{M}(t)$ is a discrete weak form of heat equation. We also know that Q^* is concentrated on π a.c measures w.r.t. to Lebesgue.

Moreover the weak form of heat equation with initial condition $\int_{\mathbb{T}^d} \rho(0,x) f(x) = \int_{\mathbb{T}^d} dx\, \rho_0(x) f(x)$ is unique, and also a strong solution.

$\Rightarrow$ ⊙ $Q_{N_k} \xrightarrow[N_k]{(w)} \delta_{\rho(t,x)dx}$, i.e.

every converging subsequence converges to the deterministic measure concentrated on the unique sol. of the heat equation $\begin{cases} \partial_t \rho = \frac{1}{2} \partial_{xx} \rho \\ \rho(0,x) = \rho_0(x) \end{cases}$.

The statement of Prop. 36 asks convergence in probability, here we have a weak convergence.

But ⊙ implies convergence in probability for the random variable $\pi_t^N(du)$

Because of the following theorem:

Prop.

Given a sequence $(X_n, \Omega_n, \mathcal{F}_n)$, with (S,d) metric space and $X_n : \Omega_n \to S$, s.t. $X_n \xrightarrow[n]{(w)} a \in S$ where $a \in S$ is a deterministic value, then

$$\mathbb{P}_n \left(d(X_n, a) < \varepsilon \right) \xrightarrow[n]{} 1.$$ Namely

$X_n \xrightarrow[n]{(w)} a$ implies $X_n \xrightarrow[n]{\mathbb{P}} a$.

Proof take $f = \chi_G$ where χ_G is the characteristic function of the open set

$$G = \{ x \in S : d(x, a) < \varepsilon \}$$

$$\mathbb{E}_n \left(\chi_G (X_n) \right) \xrightarrow[n]{} \mathbb{E} \left(\chi_G (a) \right) = 1$$

$$\| $$

$$\int_{\Omega_n} \chi_G (X_n) \, d\mathbb{P}_n = \mathbb{P}_n (X_n \in G)$$

$\blacksquare$

Remark