Hydrodynamic limit for the exclusion process on the circle via relative entropy method

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Chapter 1

The relative entropy method

In this chapter we will introduce Yau's relative entropy inequality and we will explain how the relative entropy method works.

1.1 Relative entropy

Firstly, we need to explain what is the relative entropy. For, that purpose,

Definition 1.1.1. *Let μ and ν be two probability measures in the same state space Ω. The relative entropy of μ with respect to ν is defined as*

$$H\left(\mu \,|\, \nu\right) := \sup_{f} \left\{ \int f(\eta)\, \mu(d\,\eta) - \log \int e^{f(\eta)} \nu\left(d\,\eta\right) \right\}, \tag{1.1.1}$$

where the supreme is considered over all continuous functions $f : \Omega \to \mathbb{R}$.

By the definition of relative entropy, we can observe the following inequality:

Proposition 1.1.2 (Entropy inequality)**.** *Let $B > 0$. Let μ and ν be two probability measures in Ω. Let $f : \Omega \to \mathbb{R}$ be any continuous function. We have*

$$\int f\left(\eta\right) \mu(d\,\eta) \leq \frac{1}{B} \left(H\left(\mu \,|\, \nu\right) + \log \int e^{\{B\, f\,(\eta)\}}\, \nu\left(d\,\eta\right) \right).$$

An immediate consequence of the Entropy inequality is the following:

Corollary 1.1.3. *Let $A \subset \Omega$. Let μ and ν be two probability measures in Ω. We have*

$$\mu(A) \leq \frac{\log 2 + H(\mu \,|\, \nu)}{\log(1 + 1/\nu(A))}.$$

Proof. Take $f = \mathbb{1}_A$ and $B = \log(1 + 1/\nu(A))$. $\square$

Proposition 1.1.4. *If μ is a measure absolutely continuous with respect to ν, then*

$$H(\mu \,|\, \nu) = \int f(\eta) \, \log f(\eta) \, \nu(d\eta), \tag{1.1.2}$$

where f is the Radom-Nykodym derivative of μ with respect to ν, that is,

$$f(\eta) = \frac{\mu(\eta)}{\nu(\eta)}.$$

For more properties concerning the relative entropy we suggest the reading of [2].

1.2 Yau's relative entropy inequality

In [**?**], Yau proved a famous inequality concerning relative entropy. We will show a proof of this inequality which is very different from Yau's one and it was found in [3]. In order to do so, let us consider a continuous-time Markov chain $\{X_t\,;\, t \geq 0\}$ with state space Ω and generator $\mathfrak{L}$ which acts on function $f : \Omega \to \mathbb{R}$ by

$$\mathfrak{L} f(\eta) = \sum_{\xi \in \Omega} r(\eta, \xi) \, (f(\xi) - f(\eta)),$$

where $r(\eta, \xi)$ stands for the transition rate from state η to state ξ. Let S_t denote the semigroup associated to $\mathfrak{L}$.

Definition 1.2.1. *We say that a measure ν in Ω is a* reference measure *if $\nu(x) > 0$ for any $x \in \Omega$.*

Definition 1.2.2. *Fix a reference measure ν and fix $T > 0$. Let $\{\nu_t; t \in [0, T]\}$ be a family of reference measures in Ω, differentiable with respect to t. Let $\psi_t : \Omega \to [0, \infty)$ be the Radon-Nikodym derivative of ν_t with respect to ν, that* *is,* $\psi_t(x) = \frac{\nu_t(x)}{\nu(x)}$ *for any* $t \in [0,T]$ *and any* $x \in \Omega$*. Let* $\mathfrak{L}_t^*$ *be the adjoint of* $\mathfrak{L}$ *with respect to* ν_t*. The action of* $\mathfrak{L}_t^*$ *over a function* $g : \Omega \to \mathbb{R}$ *is given by*

$$\mathfrak{L}_t^* \, g(\eta) = \sum_{\xi \in \Omega} \Big\{ r(\xi,\eta) g(\xi) \frac{\mu(\xi)}{\mu(\eta)} - r(\eta,\xi) g(\eta) \Big\}. \tag{1.2.1}$$

Remark 1.2.3. *In general,* ν_t *will* not *be an invariant measure of* $\{X_t \,;\, t \geq 0\}$ *and therefore* $\mathfrak{L}_t^*$ *will not be necessarily a Markovian operator.*

Definition 1.2.4. *The carré du champ* Γ *operator associated to* $\mathfrak{L}$ *is defined as*

$$\Gamma \, f\,(\eta) = \sum_{\xi \in \Omega} r\,(\eta,\xi)\,(f(\eta) - f(\xi))^2,$$

for any function $f : \Omega \to \mathbb{R}$*.*

Proposition 1.2.5 (Yau's inequality)**.** *Let* μ *denote the initial measure of* $\{X_t \,;\, t \geq 0\}$*. Let* $f_t : \Omega \to [0,\infty)$ *be the Radon-Nydokym derivative of the law of* $\{X_t \,;\, t \geq 0\}$ *and* ν_t*, that is,*

$$f_t(\eta) := \frac{\mu \, S_t\,(\eta)}{\nu_t\,(\eta)}, \text{ for any } \eta \in \Omega \text{ and any } t \in [0,T].$$

Define $H\,(t) := H\,(\mu \, S_t \,|\, \nu_t)$*. For any* $t \in [0,T]$*,*

$$\partial_t \, H\,(t) \leq - \int \Gamma \sqrt{f_t} d\nu_t + \int \big(\mathfrak{L}_t^* \mathbb{1} - \partial_t \log \psi_t \big) \, f_t \, d\nu_t.$$

Proof. Let $\mathfrak{L}^*$ be the adjoint of $\mathfrak{L}$ with respect to the reference measure ν. The forward Fokker-Planck equation asserts that

$$\partial_t \big(f_t \, \psi_t \big) = \mathfrak{L}^* \big(f_t \, \psi_t \big)$$

for any $t \in [0,T]$, from where

$$\partial_t f_t = \frac{1}{\psi_t} \Big(\mathfrak{L}^* \big(f_t \psi_t \big) - f_t \, \partial_t \, \psi_t \Big).$$

Therefore, rewriting $H(t)$ as $H(t) = \int f_t \log f_t \psi_t d\nu$, we see that

$$
\begin{aligned}
\partial_t H(t) &= \int (1 + \log f_t)\big(\mathfrak{L}^*\big(f_t\psi_t\big) - f_t\partial_t\psi_t\big) d\nu \\
&\quad + \int f_t \log f_t \partial_t \psi_t d\nu \\
&= \int f_t \mathfrak{L} \log f_t d\nu_t - \int f_t\, \partial_t \log \psi_t d\nu_t.
\end{aligned}
$$

Now, since $a(\log b - \log a) \leq 2\sqrt{a}(\sqrt{b} - \sqrt{a})$, we obtain

$$
\begin{aligned}
f_t(\eta)\mathfrak{L} \log f_t(\eta) &= \sum_{\xi\in\Omega} r(\eta,\xi) f_t(\eta)\big(\log f_t(\xi) - \log f_t(\eta)\big) \\
&\leq \sum_{\xi\in\Omega} 2r(\eta,\xi)\sqrt{f_t(\eta)}\big(\sqrt{f_t(\xi)} - \sqrt{f_t(\eta)}\big)
\end{aligned}
$$

for any $\eta \in \Omega$. Moreover, since $2\sqrt{a}(\sqrt{b} - \sqrt{a}) = -(\sqrt{b} - \sqrt{a})^2 + b - a$, we conclude that

$$
2\, r(\eta,\xi)\sqrt{f_t(\eta)}\big(\sqrt{f_t(\xi)} - \sqrt{f_t(\eta)}\big) = -r(\eta,\xi)\big(\sqrt{f_t(\xi)} - \sqrt{f_t(\eta)}\big)^2 + r(\eta,\xi)\big(f_t(\xi) - f_t(\eta)\big).
$$

Therefore, $2\sqrt{f_t}\,\mathfrak{L}\sqrt{f_t} = -\Gamma\sqrt{f_t} + \mathfrak{L} f_t$. Hence, we obtain that

$$
\partial_t H(t) \leq -\int \Gamma\sqrt{f_t} d\nu_t + \int \Big(\mathfrak{L} f_t - f_t\, \partial_t \log \psi_t\Big) d\nu_t,
$$

which implies the desired inequality due to the fact that $\int \mathfrak{L} f_t d\nu_t = \int L_t^* \mathbb{1} f_t d\nu_t$.

□

Exercise 1.2.6. *Show that $\int f_t\, \partial_t \log \psi_t\, d\nu_t$ is constant on ν.*

Exercise 1.2.7. *Show that if ν_t is the invariant measure of $\{X_t\,;\, t \geq 0\}$ then*

$$
\int \Big(\mathfrak{L} f_t - f_t\, \partial_t \log \psi_t\Big) d\nu_t = 0. \tag{1.2.2}
$$

In particular,

$$
\partial_t H(t) \leq -\int \Gamma\sqrt{f_t} d\nu_t. \tag{1.2.3}
$$

Here, one can that see the the term on the left-hand side of (1.2.2) is the price that they have to pay when changing the Dirichlet form (carré du champ asso-

ciated with the invariant measure) to the carré du champ associated with the measure ν_t*.*

1.3 Yau's relative entropy method

Now we explain the relative entropy method [4]. The idea is simple. In many problems, one wants to show that two measures μ and ν on the same probability space are close, in some distance. Although the relative entropy is not actually a metric, it is commonly used as one. For instance, the relative entropy $H(\mu\,|\,\nu)$ bounds the total variation distance $\|\mu-\nu\|_{TV}$ from above (Pinsker's inequality) and when μ is the time distribution of the process and ν is invariant, the combination of (1.2.3) with log-Sobolev inequalities $H(\mu\,|\,\nu)\leq C\,\Gamma\sqrt{f_t}d\nu$ shows strong properties on the mixing time of this Markov chain. As we will see in the next chapter, in order to prove the hydrodynamic limit of a particle system, we will show choose reference measures ν_t which are associated with this hydrodynamic equation and we will show that the relative entropy of the time distribution of the process (starting at nice initial measures) and ν_t keeps bounded by a small enough sequence in $n\in\mathbb{N}$. This choice on the reference measures and the bounds obtained on the relative entropy are explained by the conservation of local equilibrium. Moreover, the relative entropy method requires the existence of a smooth solution of such PDE's and it implies the uniqueness itself.

Now one may ask how can we use the hydrodynamic equation to prove the hydrodynamic limit whether we do not know which PDE should it be. As we will see, the computation of the left-hand side of (1.2.2) will be given in a nice algorithm found in [3], and it presents a closed expression. In order to close this expression, we will see which PDE should we consider, as well as which boundary conditions and initial data.

Chapter 2

Hydrodynamic limits of interacting particle systems

2.1 Simple exclusion process

Let $G = (V, E)$ be an undirected graph with vertex set V, edge set E and order n. In this model, each vertex of V is occupied or not by a particle. The resulting matching between the vertices and their occupations is called a *configuration* of particles and it can be seen as a function $\eta : V \to \{0, 1\}$. In this description, we say that the vertex x is occupied by a particle if $\eta(x) = 1$ and that it is empty if $\eta(x) = 0$. We denote by

$$\Omega_n^k = \{\eta : V \to \{0,1\}; \sum_{x \in V} \eta(x) = k\} \tag{2.1.1}$$

the set of all configurations with exactly k particles. The *simple exclusion process* (SEP) in G is described as follows: Exponential clocks of rate 1 are attached, independently, to each edge in E. Whenever a clock rings, the edge associated with that clock is flipped, exchanging the occupations of its incident vertices. The probability that two clocks ring at the same time is zero. Since particles are indistinguishable, we can suppose that when both vertices incident to the same edge have the same occupation, the interaction between those particles (if the vertices are occupied) does not happen. This is called *exclusion rule*. Note that the dynamics conserves the total mass, that is, the number of particles in the system does not change. Therefore, if the initial configuration has exactly k

particles, then the SEP in G is the continuous-time Markov process $\{\eta_t\,;\, t \geq 0\}$ with state space Ω_n^k whose generator is given by

$$\mathfrak{L}\, f(\eta) = \frac{1}{2} \sum_{x \in V} \sum_{y \sim V} \Big(f(\eta^{x,y}) - f(\eta) \Big), \tag{2.1.2}$$

for every function $f : \Omega_n^k \to \mathbb{R}$, where $\eta^{x,y}$ is the configuration obtained from η after exchanging the occupations of vertices x and y, that is,

$$\eta^{x,y}(z) = \begin{cases} \eta(x), & \text{if } z = y; \\ \eta(y), & \text{if } z = x; \\ \eta(x). & \text{if } z \notin \{x, y\}. \end{cases} \tag{2.1.3}$$

In the above definition, we write $x \sim y$ to say that there exists the edge $\{x, y\} \in E$. The constant $1/2$ in the right-hand side of (2.1.2) appears owing to the fact that each edge $\{x, y\}$ is being counted twice in the sums.

2.2 Hydrodynamic limit for the SEP on the torus

In this section we show how to prove the so called hydrodynamic limit using Yau's relative entropy method. We will illustrate the idea with the simple exclusion process on the discrete one-dimensional torus of length n (the n-circle). Indeed, let $\mathbb{T}_n = \mathbb{Z}_n = \mathbb{Z}/n\,\mathbb{Z}$ be the set of vertices of the aforementioned graph. Two vertices x and y are adjacent if $x = y \pm 1 \mod n$. Define $\Omega_n := \cup_{k=0}^{n} \Omega_n^k$. The simple exclusion process on the discrete one-dimensional torus of length n is the Markov process $\{\eta_t\,;\, t \geq 0\}$ with state space Ω_n and generator

$$n^2\, \mathfrak{L}_n\, f\,(\eta) = \sum_{x \in \mathbb{T}_n} (f\,(\eta^{x,x+1}) - f\,(\eta)). \tag{2.2.1}$$

The factor n^2 in front of the generator speeds up time to the diffusive time scale. Let $D_{\Omega_n}[0, T]$ be the path space of cádlág time trajectories with values in Ω_n, which is called *Skorohod space*. We will denote by $\mathbb{P}_{\mu_n}$ the probability measure on $D_{\Omega_n}[0, T]$ induced by the initial measure μ_n and the Markov process $\{\eta_t; t \geq 0\}$. The expectation with respect to $\mathbb{P}_{\mu_n}$ will be denoted by $\mathbb{E}_{\mu_n}$. We will also denote the semigroup associate with $\mathfrak{L}_n$ by S_t and we will define $\mu_t := \mu_n\, S_t$.

Let $\mathbb{T}$ denote the one dimensional torus. Let us fix a profile $\rho_0 : \mathbb{T} \to [\varepsilon_0, 1 - \varepsilon_0]$ for some $\varepsilon_0 \in (0, 1/2]$ and a time horizon $[0, T]$, $T > 0$. Let $\rho :$

$[0,T] \times \mathbb{T} \to [\varepsilon_0, 1-\varepsilon_0]$ be a smooth solution of

$$\partial_t \, \rho\,(t,u) = \Delta\, \rho\,(t,u),\; u \in \mathbb{T}\,,\; t \in [0,T].$$

Let

$$\rho_t^n(x) = \rho\left(t, \frac{x}{n}\right).$$

For each $t \in [0,T]$ we define the measure ν_t on Ω_n as the Bernoulli product measure

$$\nu_t\,(\eta) = \prod_{x=1}^{n-1} \left\{\eta(x)\, \rho_t^n\,(x) + (1-\eta(x))\; (1-\rho_t^n\,(x))\right\}.$$

We will prove the following results:

Teorema 2.2.1. *If $H\,(\mu_n \,|\, \nu_0) = \mathcal{O}(1)$ then $H\,(\mu_t \,|\, \nu_t) = \mathcal{O}(1)$.*

Corollary 2.2.2 (Hydrodynamic limit)**.** *For any $\delta > 0$, any $t \in [0,T]$ and any $H \in C\,(\mathbb{T})$, let us define the stochastic process*

$$A_{t,H} := \frac{1}{n} \sum_{x \in \mathbb{T}_n} \eta_t(x)\, H\left(\frac{x}{n}\right) - \int_{\mathbb{T}} \rho\,(t,u)\, H\,(u)\, d\,u \tag{2.2.2}$$

and the event

$$A_{t,\delta,H} := \{\eta\,;\, |A_{t,H}| > \delta\} \tag{2.2.3}$$

If $H\,(\mu_n \,|\, \nu_0) = \mathcal{O}(1)$ then for any $\delta > 0$, any $t \in [0,T]$ and any $H \in C\,(\mathbb{T})$

$$\lim_{n\to\infty} \mathbb{P}_{\mu_n}\,(A_{t,\delta,H}) = 0.$$

We first prove the corollary. We will use Hoeffding's Lemma, whose proof can be found in Wikipedia.

Exercise 2.2.3 (Hoeffding's Lemma)**.** *Let η be a random value taking values on $[0,1]$. If $m = \mathbb{E}[\eta]$ then for any $\theta \in \mathbb{R}$ we have*

$$\log \mathbb{E}\left[e^{\theta\,(\eta - m)}\right] \le \frac{\theta^2}{8}.$$

Proof of Corollary 2.2.2. We give a complete proof of this result based in the sketch of the proof of [1, Theorem 2.9.1]. However, we do not use large deviation estimates and that why we require Hoeffding's Lemma. Firstly, observe that, by Corollary 1.1.3, we have

$$\mathbb{P}_{\mu_n}\left(A_{t,\delta,H}\right) \leq \frac{\log 2 + H\left(\mu_t \,|\, \nu_t\right)}{\log\left(1 + 1/\nu_t\left(A_{t,\delta,H}\right)\right)}. \tag{2.2.4}$$

Now, let us define the stochastic process

$$B_{t,H} := \frac{1}{n} \sum_{x \in \mathbb{T}_n} \left(\eta_t(x) - \rho_t^n(x)\right) H\left(\frac{x}{n}\right). \tag{2.2.5}$$

and the sequence

$$C_{t,H} := \frac{1}{n} \sum_{x \in \mathbb{T}_n} \rho_t^n(x)\, H\left(\frac{x}{n}\right) - \int_{\mathbb{T}} \rho\left(t,u\right) H\left(u\right) d\,u.$$

Thus,

$$\begin{aligned} \nu_t\,\left(A_{t,\delta,H}\right) &= \nu_t\,\left(A_{t,H} > \delta\right) + \nu_t\,\left(A_{t,-H} > \delta\right) \\ &= \nu_t\,\left(B_{t,H} + C_{t,H} > \delta\right) + \nu_t\,\left(B_{t,-H} + C_{t,-H} > \delta\right). \end{aligned}$$

Since $\frac{1}{n}\sum_{x\in\mathbb{T}_n} \rho_t^n(x)\, H\left(\frac{x}{n}\right)$ converges to $\int_{\mathbb{T}} \rho\left(t,u\right) H\left(u\right) d\,u$, for sufficiently large n we have

$$-\frac{\delta}{2} < C_{t,H} < \frac{\delta}{2}$$

Therefore, $\nu_t\,\left(A_{t,\delta,H}\right) \leq \nu_t\,\left(B_{t,H} > \delta/2\right) + \nu_t\,\left(B_{t,-H} > \delta/2\right)$. Moreover, by Chebyshev's exponential inequality $\boxed{\mathbb{P}(X > b) \leq e^{-a\,b}\,\mathbb{E}[e^{a\,X}]}$ and since ν_t is a product measure, for any $a > 0$ we have

$$
\begin{aligned}
\nu_t\,(A_{t,\delta,H}) \le e^{-a\,\delta/2}\mathbb{E}[e^{a\,B_{t,H}}] + e^{-a\,\delta/2}\mathbb{E}[e^{a\,B_{t,-H}}]&\\
= e^{-a\,\delta/2}\ \nu_t\left[\prod_{x\in\mathbb{T}_n}\exp\left\{\frac{a\,H\left(\frac{x}{n}\right)}{n}\,(\eta_t(x)-\rho_t^n(x))\right\}\right]&\\
+\, e^{-a\,\delta/2}\ \nu_t\left[\prod_{x\in\mathbb{T}_n}\exp\left\{-\frac{a\,H\left(\frac{x}{n}\right)}{n}\,(\eta_t(x)-\rho_t^n(x))\right\}\right]&\\
= e^{-a\,\delta/2}\ \prod_{x\in\mathbb{T}_n}\nu_t\left[\exp\left\{\frac{a\,H\left(\frac{x}{n}\right)}{n}\,(\eta_t(x)-\rho_t^n(x))\right\}\right]&\\
+\, e^{-a\,\delta/2}\ \prod_{x\in\mathbb{T}_n}\nu_t\left[\exp\left\{-\frac{a\,H\left(\frac{x}{n}\right)}{n}\,(\eta_t(x)-\rho_t^n(x))\right\}\right]&
\end{aligned}
$$

$$
\begin{aligned}
&= e^{-a\,\delta/2}\ \exp\left\{\log\prod_{x\in\mathbb{T}_n}\nu_t\left[\exp\left\{\frac{a\,H\left(\frac{x}{n}\right)}{n}\,(\eta_t(x)-\rho_t^n(x))\right\}\right]\right\}\\
&+ e^{-a\,\delta/2}\ \exp\left\{\log\prod_{x\in\mathbb{T}_n}\nu_t\left[\exp\left\{-\frac{a\,H\left(\frac{x}{n}\right)}{n}\,(\eta_t(x)-\rho_t^n(x))\right\}\right]\right\}\\
&= e^{-a\,\delta/2}\ \exp\left\{\sum_{x\in\mathbb{T}_n}\log\,\nu_t\left[\exp\left\{\frac{a\,H\left(\frac{x}{n}\right)}{n}\,(\eta_t(x)-\rho_t^n(x))\right\}\right]\right\}\\
&+ e^{-a\,\delta/2}\ \exp\left\{\sum_{x\in\mathbb{T}_n}\log\,\nu_t\left[\exp\left\{-\frac{a\,H\left(\frac{x}{n}\right)}{n}\,(\eta_t(x)-\rho_t^n(x))\right\}\right]\right\}.
\end{aligned}
$$

Now, by Hoeffding's Lemma (see Exercise 2.2.3) we have

$$
\begin{aligned}
\nu_t\,(A_{t,\delta,H}) &\le 2\,e^{-a\,\delta/2}\ \exp\left\{\frac{a^2}{n^2}\sum_{x\in\mathbb{T}_n}H^2\left(\frac{x}{n}\right)\right\}\\
&= \exp\left\{\frac{a^2}{n}\|H\|_\infty^2+\log 2-\frac{a\,\delta}{2}\right\}.
\end{aligned}
$$

Choosing $a=\frac{\delta\,n}{4\,\|H\|_\infty^2}$ we obtain that

$$
\nu_t\,(A_{t,\delta,H}) \le \exp\left\{-\frac{\delta^2}{16\,\|H\|_\infty^2}\,n+\log 2\right\}
$$

Hence, there exists a constant $C = C\,(\delta)$ such that for any sufficiently large n

we have

$$\nu_t\left(A_{t,\delta,H}\right) < e^{-C\,n}. \tag{2.2.6}$$

Last, replacing (2.2.6) in (2.2.4) we obtain

$$\mathbb{P}_{\mu_n}\left(A_{t,\delta,H}\right) < \frac{\log 2 + H\left(\mu_t\,|\,\nu_t\right)}{C\,n},$$

which converges to zero by Theorem 2.2.1. □

Now, let Δ_n and ∇_n stand for the discrete Laplacian and gradient, respectively defined as

$$\Delta_n\, f(x) = n^2\left(f(x+1) + f(x-1) - 2\,f(x)\right)$$

and

$$\nabla_n\, f(x) = n\left(f(x+1) - f(x)\right).$$

Recall that we want to prove Theorem 2.2.1. Before we do so, we show the validity of the following lemma:

Lemma 2.2.4. *For each $x \in \mathbb{T}_n$ define*

$$\omega_t(x) = \frac{\eta(x) - \rho_t^n(x)}{\rho_t^n(x)\left(1 - \rho_t^n(x)\right)}.$$

Let $H(t) = H\left(\mu_t\,|\,\nu_t\right)$. There exists a constant $C = C(\varepsilon_0)$ such that

$$\begin{aligned}\partial_t\, H\left(t\right) \le &-\int \Gamma\sqrt{f_t} d\nu_t + H(t) + C\\ &- \sum_{x\in\mathbb{T}_n}\left(\nabla_n\,\rho_t^n(x)\right)^2 \mathbb{E}_{\mu_n}\,\left[\omega_t(x)\,\omega_t(x+1)\right].\end{aligned}$$

Proof. Recall Yau's inequality (Proposition 1.2.5),

$$\partial_t\, H\left(t\right) \le -\int \Gamma\sqrt{f_t} d\nu_t + \int\left(\mathfrak{L}_t^*\mathbb{1} - \partial_t \log\psi_t\right) f_t\, d\nu_t.$$

First let us compute $\mathfrak{L}_t^* \mathbb{1}$. Indeed, by identity (1.2.1) we have

$$
\begin{aligned}
\mathfrak{L}_t^*\, \mathbb{1}(\eta) &= n^2 \sum_{x\in\mathbb{T}_n} \Big\{ \eta(x)\,(1-\eta(x+1))\frac{\mu(\eta^{x,x+1})}{\mu(\eta)} - \eta(x+1)\,(1-\eta(x)) \Big\} \\
&+ n^2 \sum_{x\in\mathbb{T}_n} \Big\{ \eta(x+1)\,(1-\eta(x))\frac{\mu(\eta^{x,x+1})}{\mu(\eta)} - \eta(x)\,(1-\eta(x+1)) \Big\} \\
&= n^2 \sum_{x\in\mathbb{T}_n} \Big\{ \eta(x)\,(1-\eta(x+1))\frac{\rho_t^n(x+1)\,(1-\rho_t^n(x))}{\rho_t^n(x)\,(1-\rho_t^n(x+1))} - \eta(x+1)\,(1-\eta(x)) \Big\} \\
&+ n^2 \sum_{x\in\mathbb{T}_n} \Big\{ \eta(x+1)\,(1-\eta(x))\frac{\rho_t^n(x)\,(1-\rho_t^n(x+1))}{\rho_t^n(x+1)\,(1-\rho_t^n(x))} - \eta(x)\,(1-\eta(x+1)) \Big\}\,.
\end{aligned}
$$

Therefore, factorizing the above identity, we obtain

$$
\begin{aligned}
\mathfrak{L}_t^*\, \mathbb{1}(\eta) = n^2 &\sum_{x\in\mathbb{T}_n} (\rho_t^n(x) - \rho_t^n(x+1)) \times \\
&\times \Big(\frac{\eta(x+1)\,(1-\eta(x))}{\rho_t^n(x+1)\,(1-\rho_t^n(x))} - \frac{\eta(x)\,(1-\eta(x+1))}{\rho_t^n(x)\,(1-\rho_t^n(x+1))} \Big) \qquad (2.2.7)
\end{aligned}
$$

Recall that

$$
\omega_t(x) = \frac{\eta(x)-\rho_t^n(x)}{\rho_t^n(x)\,(1-\rho_t^n(x))}.
$$

We will write (2.2.7) as linear combination of 1, $\omega_t(x)$, $\omega_t(x+1)$ and $\omega_t(x)\,\omega_t(x+1)$, that is, we will find $a, b, c, d \in \mathbb{R}$ such that

$$
\begin{aligned}
\Big(\frac{\eta(x+1)\,(1-\eta(x))}{\rho_t^n(x+1)\,(1-\rho_t^n(x))} - \frac{\eta(x)\,(1-\eta(x+1))}{\rho_t^n(x)\,(1-\rho_t^n(x+1))} \Big) &= a + b\,\omega_t(x) + c\,\omega_t(x+1) \\
&+ d\,\omega_t(x)\,\omega_t(x+1). \quad (2.2.8)
\end{aligned}
$$

Indeed, taking the expectation in both sides of (2.2.8), with respect to ν_t we obtain that $a = 0$. Evaluating (2.2.8) at $\eta(x) = 1$ and $\eta(x+1) = \rho_t^n(x+1)$ (it is the same as taking the expectation with respect to $\mathrm{Bern}(1) \times \mathrm{Bern}(\rho_t^n(x))$), we obtain that $b = -1$. Taking $\eta(x) = \rho_t^n(x)$ and $\eta(x+1) = 1$, we obtain $c = 1$. Last, taking $\eta(x) = \eta(x+1) = 1$ we obtain the relation

$$
\frac{d}{\rho_t^n(x)\,\rho_t^n(x+1)} = \frac{1}{\rho_t^n(x)}\frac{1}{\rho_t^n(x+1)}.
$$

Therefore, $d = (\rho_t^n(x+1) - \rho_t^n(x))$ and hence, a summation by parts shows that

$$\mathfrak{L}_t^* \,\mathbb{1}(\eta) = \sum_{x\in\mathbb{T}_n} (\Delta_n \,\rho_t^n\,(x))\,\omega_t(x) - \sum_{x\in\mathbb{T}_n} (\nabla_n\,\rho_t^n(x))^2\,\omega_t(x)\,\omega_t(x+1). \quad (2.2.9)$$

Now let us compute $\partial_t \log \psi_t$. Indeed, let the reference measure ν be the Bernoulli product measure with parameters $1/2$ (recall that we can choose any reference measure). Since ν_t is a product measure, we have

$$\begin{aligned}
\partial_t \log \psi_t^n &= \partial_t \sum_{x\in\mathbb{T}_n} \big(\eta(x)\log(2\,\rho_t^n(x)) + (1-\eta(x))\log(2\,(1-\rho_t^n(x)))\big) \\
&= \sum_{x\in\mathbb{T}_n} \big(\eta(x)\partial_t\, \log(2\,\rho_t^n(x)) + (1-\eta(x))\partial_t\, \log(2\,(1-\rho_t^n(x)))\big) \\
&= \sum_{x\in\mathbb{T}_n} \Big(\frac{\eta(x)}{\rho_t^n(x)} - \frac{1-\eta(x)}{1-\rho_t^n(x)}\Big)\partial_t\,\rho_t^n(x) \\
&= \sum_{x\in\mathbb{T}_n} \omega_t(x)\,\partial_t\,\rho_t^n(x). \qquad (2.2.10)
\end{aligned}$$

Therefore, by Yau's inequality (Proposition 1.2.5), and identities (2.2.9) and (2.2.10), we obtain that

$$\begin{aligned}
\partial_t\, H\,(t) \le -\int \Gamma\sqrt{f_t}d\nu_t &+ \sum_{x\in\mathbb{T}_n} \mathbb{E}_{\mu_n}\,\left[\omega_t(x)\right]\big(\Delta_n - \partial_t\big)\rho_t^n(x) \\
&- \sum_{x\in\mathbb{T}_n} (\nabla_n\,\rho_t^n(x))^2\,\mathbb{E}_{\mu_n}\,\left[\omega_t(x)\,\omega_t(x+1)\right].
\end{aligned}$$

Furthermore, since Δ_n is a discrete approximation or order $\mathcal{O}(n^{-2})$ of Δ (for any f of class $\mathcal{C}^4$, $\sup_x |(\Delta_n - \Delta)\,f\,(x)| < C(f)\,n^{-2}$), the above inequality is of the form

$$\begin{aligned}
\partial_t\, H\,(t) \le -\int \Gamma\sqrt{f_t}d\nu_t &+ \mathbb{E}_{\mu_n}\,\left[\frac{1}{n^2}\sum_{x\in\mathbb{T}_n} \omega_t(x)\,R_t(x)\right] \\
&- \sum_{x\in\mathbb{T}_n} (\nabla_n\,\rho_t^n(x))^2\,\mathbb{E}_{\mu_n}\,\left[\omega_t(x)\,\omega_t(x+1)\right], \qquad (2.2.11)
\end{aligned}$$

where $|R_t(x)| \le |\rho_t|_{\mathcal{C}^4}$. Furthermore, by Proposition 1.1.2 and Hoeffding's

Lemma (see Exercise 2.2.3), we have

$$\mathbb{E}_{\mu_n}\left[\frac{1}{n^2}\sum_{x\in\mathbb{T}_n}\omega_t(x)\,R_t(x)\right] \le H(t)+\log\int\exp\left\{\frac{1}{n^2}\sum_{x\in\mathbb{T}_n}\omega_x R^n_x(t)\right\}d\mu^n_t$$
$$\le H(t)+\frac{C(\varepsilon_0)}{n^3}\|R^n(t)\|^2_\infty$$
$$\le H(t)+C\,(\varepsilon_0).$$

Hence,

$$\partial_t\,H\,(t)\le-\int\Gamma\sqrt{f_t}d\nu_t+H(t)+C\,(\varepsilon_0)$$
$$-\sum_{x\in\mathbb{T}_n}(\nabla_n\,\rho^n_t(x))^2\,\mathbb{E}_{\mu_n}\,[\omega_t(x)\,\omega_t(x+1)]\,.$$

□

Now we state the following result which is proven in [3] for a more general setting:

Lemma 2.2.5 (Replacement Lemma)**.** *Recall that there exists $\kappa>0$ such that $\nabla_n\,\rho^n_t(x)\le\kappa$ for any $x\in\mathbb{T}_n$. There exists a finite constant $C=C(\varepsilon_0)$ such that for any $G:\mathbb{T}_n\to\mathbb{R}$ and any $\delta>0$ we have*

$$\sum_{x\in\mathbb{T}_n}G(x)\,\mathbb{E}_{\mu_n}\,[\omega_t\,(x)\,\omega_t\,(x+1)]\le\delta\int\Gamma\sqrt{f}\,\nu_t\,(d\,\eta)$$
$$+\frac{C\,(1+\kappa^2)}{\delta}\,(\|G\|_\infty+\|G\|^2_\infty)\;(H\,(t)+1)\,.$$

Proof of Theorem 2.2.1. By Lemma 2.2.4 and Lemma 2.2.5, with

$$G_t(x) = (\nabla_n \, \rho_t^n \, (x))^2,$$

there exists a constant $C = C\,(\varepsilon_0)$ such that for any $\delta > 0$ we have

$$\begin{aligned}
\partial_t \, H\,(t) &\leq -(1-\delta) \int \Gamma \sqrt{f_t} d\nu_t + H(t) + C\,(\varepsilon_0) \\
&+ \frac{C\,(1+\kappa^2)}{\delta} \left(\|(\nabla_n \, \rho_t^n)^2\|_\infty + \|(\nabla_n \, \rho_t^n)^2\|_\infty^2 \right) (H\,(t) + 1) \\
&= -(1-\delta) \int \Gamma \sqrt{f_t} d\nu_t + H(t) + C\,(\varepsilon_0) \\
&+ \frac{C\,(1+\kappa^2)^2\,\kappa^2}{\delta} \, (H\,(t) + 1)\,.
\end{aligned}$$

The proof finishes taking $\delta \in (0, 1]$ and using Grönwall's inequality. $\square$

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