On the asymptotic behavior of slowed exclusion processes

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8th March 2016
The processes

Slowed exclusion processes: the dynamics

- \( \eta_t \) is an exclusion process with space state \( \Omega = \{0, 1\}^\mathbb{Z} \), so that for \( x \in \mathbb{Z}, \eta(x) = 1 \) if the site is occupied, otherwise \( \eta(x) = 0 \).

The rates are:

We assume \( \gamma > \beta \) or \( \beta = \gamma \) and \( \alpha \geq a \) (in last case if \( a = \alpha \) then \( \{-1, 0\} \) is totally asymmetric).

- For \( a = 0 \), we obtain the SSEP with a slow bond.
- For \( \alpha = 1 \) and \( \beta = 0 \) we obtain the WASEP - weak asymmetry.
- \( \nu_\rho \) the Bernoulli product measure of parameter \( \rho \) is invariant.
Hydrodynamic limit: the case $a = 0$

- For $\eta$ let $\pi^n_t(\eta; du) = \frac{1}{n} \sum_{x \in \mathbb{Z}} \eta x^2(x) \delta_{x/n}(du)$.

- Fix $\rho_0 : \mathbb{R} \to [0, 1]$ and $\mu_n$ such that for every $\delta > 0$ and every continuous function $H : \mathbb{R} \to \mathbb{R}$,

$$\frac{1}{n} \sum_{x \in \mathbb{Z}} H\left(\frac{x}{n}\right) \eta(x) \to_{n \to \infty} \int_{\mathbb{R}} H(u) \rho_0(u) du,$$

wrt $\mu_n$. Then for any $t > 0$, $\pi^n_t \to \rho(t, u) du$, as $n \to \infty$, where $\rho(t, u)$ evolves according to:

- $\beta < 1$: Heat equation $\partial_t \rho(t, u) = \Delta \rho(t, u)$

- $\beta = 1$: Heat equation $\partial_t \rho(t, u) = \Delta \rho(t, u)$ with a type of **Robin’s** boundary conditions $\partial_u \rho(t, 0^-) = \partial_u \rho(t, 0^+) = \alpha(\rho(t, 0^+) - \rho(t, 0^-))$.

- $\beta > 1$: Heat equation $\partial_t \rho(t, u) = \Delta \rho(t, u)$ with **Neumann’s** boundary conditions $\partial_u \rho(t, 0^-) = \partial_u \rho(t, 0^+) = 0$. 
Equilibrium density fluctuations: \( a = 0 \)

- Fix a density \( \rho \in (0, 1) \) and consider the process starting from \( \nu_\rho \).

- The density fluctuation field \( \{ \mathcal{Y}_{t}^{\beta,\gamma,n} ; t \in [0,T] \} \) is given on \( H \in S_\beta(\mathbb{R}) \) by

\[
\mathcal{Y}_{t}^{\beta,\gamma,n}(H) := \frac{1}{\sqrt{n}} \sum_{x \in \mathbb{Z}} H \left( \frac{x}{n} \right) (\eta_{tn^2}(x) - \rho).
\]

**Definition**

Let \( S(\mathbb{R}\backslash\{0\}) \) be the space of functions \( H : \mathbb{R} \to \mathbb{R} \) such that: 1) \( H \) is smooth on \( \mathbb{R}\backslash\{0\} \), 2) \( H \) is continuous from the right at 0, 3) for all non-negative integers \( k, \ell \), the function \( H \) satisfies

\[
\|H\|_{k,\ell} := \sup_{u \neq 0} \left| (1 + |u|^{\ell}) \frac{d^{k}H}{du^{k}}(u) \right| < \infty.
\]
Space of test functions

**Definition**

1. For $\beta < 1$, $S_\beta(\mathbb{R}) := S(\mathbb{R})$, the usual Schwartz space $S(\mathbb{R})$.
2. For $\beta = 1$, $S_\beta(\mathbb{R})$ is the subset of $S(\mathbb{R}\{0\})$ composed of functions $H$ such that
   \[
   \frac{d^{2k+1}H}{du^{2k+1}}(0^+) = \frac{d^{2k+1}H}{du^{2k+1}}(0^-) = \alpha \left( \frac{d^{2k}H}{du^{2k}}(0^+) - \frac{d^{2k}H}{du^{2k}}(0^-) \right)
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   for any integer $k \geq 0$.
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Equilibrium fluctuations
Density fluctuation field for $a = 0$

Theorem (Franco, G., Neumann - 2013)

If $a = 0$, the sequence of processes $\{Y_{t;\beta,\gamma,n} ; t \in [0,T]\}_{n \in \mathbb{N}}$ converges to the Ornstein-Uhlenbeck process given by

$$dY_t^\beta = \frac{1}{2} \Delta_\beta Y_t^\beta \, dt + \sqrt{\chi(\rho)\nabla_\beta} \, dW_t^\beta,$$

where $\{W_t^\beta ; t \in [0,T]\}$ is an $S'_\beta(\mathbb{R})$-valued Brownian motion and $\chi(\rho) = \rho(1 - \rho)$. 
Density fluctuation field for $a \neq 0$: removing the drift

We redefine for any $H \in S_\beta(\mathbb{R})$

$$\mathcal{Y}_{t,\gamma,n}^{{\beta,\gamma,n}}(H) = \frac{1}{\sqrt{n}} \sum_{x \in \mathbb{Z}} H \left( \frac{x - n^{2-\gamma}a(1 - 2\rho)t}{n} \right) (\eta_{tn^2}(x) - \rho).$$

Theorem (Ornstein-Uhlenbeck process)

If one of these two conditions are satisfied:

- $\beta \leq 1/2$ and $\gamma > 1/2$,
- $\beta > 1/2$ and $\gamma \geq \beta$

then $\{\mathcal{Y}_{t,\gamma,n}^{{\beta,\gamma,n}} ; t \in [0, T]\}_{n \in \mathbb{N}}$ converges to OU as in the case $a = 0$.

- The influence of the asymmetry is NOT SEEN in the limit.
Effect of a stronger asymmetry $a \neq 0$: the KPZ scaling

Theorem (Stochastic Burgers equation)

Fix $\rho = 1/2$. For $\beta \leq 1/2$ and $\gamma = 1/2$, $\{\mathcal{Y}_{t}^{\beta,\gamma,n} ; t \in [0,T]\}_{n \in \mathbb{N}}$ is tight and any limit point is a stationary energy solution of the stochastic Burgers equation

$$d\mathcal{Y}_{t} = \frac{1}{2} \Delta \mathcal{Y}_{t} dt + a \nabla (\mathcal{Y}_{t})^{2} dt + \sqrt{\chi(\rho)} \nabla d\mathcal{W}_{t},$$

where $\{\mathcal{W}_{t} ; t \in [0,T]\}$ is an $S'(\mathbb{R})$-valued Brownian motion.
Stochastic Burgers equation (KPZ regime)

- OU process with no boundary conditions
- OU process with Robin’s boundary conditions
- OU process with Neumann’s boundary conditions
The KPZ scaling: stationary energy solution

To show that $\mathcal{Y}_t$ is a stationary energy solution of

$$
\frac{1}{2} \Delta \mathcal{Y}_t dt + a \nabla (\mathcal{Y}_t)^2 dt + \sqrt{\chi(\rho)} \nabla d\mathcal{W}_t,
$$

we need to prove that $\{\mathcal{M}_t : t \in [0, T]\}$ given by

$$
\mathcal{M}_t(H) := \mathcal{Y}_t(H) - \mathcal{Y}_0(H) - \frac{1}{2} \int_0^t \mathcal{Y}_s(\Delta H) ds + aA_t(H)
$$

is a continuous martingale with quadratic variation

$$
\langle \mathcal{M}(H) \rangle_t = \rho(1 - \rho) \| \nabla H \|_2^2,
$$

where

$$
A_t(H) = \lim_{\varepsilon \to 0} \int_0^t \int_{\mathbb{R}} \nabla H(x) \left[ \mathcal{Y}_u(\iota_\varepsilon(x)) \right]^2 dx du
$$

in $L^2$, where $\iota_\varepsilon(x, y) = \frac{1}{\varepsilon} \mathbf{1}_{x \leq y < x + \varepsilon}$, for $y \in \mathbb{R}$. 
The instantaneous current

Note that

\[ j_n^x(x+1) = j_n^S(x+1) + j_n^A(x+1) \]

with

\[ j_n^A(x+1) = \frac{an^2}{2\gamma_n} (\eta(x+1) - \eta(x))^2, \quad x \in \mathbb{Z}, \]

\[ j_n^S(x+1) = \frac{n^2}{2} (\eta(x) - \eta(x+1)), \quad x \neq -1, \]

\[ j_{-1,0}^S(\eta) = \frac{\alpha n^2}{2\beta_n} (\eta(-1) - \eta(0)). \]
The martingale problem

Simple computations show that

$$M^n_t(H) := \mathcal{Y}^n_t(H) - \mathcal{Y}^n_0(H) - \mathcal{I}^n_t(H) - B^n_t(H),$$

plus some negligible term, where

$$\mathcal{I}^n_t(H) := \frac{1}{2} \int_0^t \mathcal{Y}^n_s(\Delta H) \, ds = \frac{1}{2} \int_0^t \frac{1}{\sqrt{n}} \sum_{x \in \mathbb{Z}} (\eta_{sn^2}(x) - \rho) \Delta H\left(\frac{x}{n}\right) \, ds,$$

and

$$B^n_t(H) = -a \frac{\sqrt{n}}{n^\gamma} \int_0^t \sum_{x \in \mathbb{Z}} \tilde{\eta}_{sn^2}(x + 1) \tilde{\eta}_{sn^2}(x) \nabla H\left(\frac{x}{n}\right) \, ds.$$

Last term is the hard one!
The second-order Boltzmann-Gibbs Principle

Theorem

Let \( v : \mathbb{Z} \to \mathbb{R} \) be a function such that \( \|v\|_{2,n}^2 := \frac{1}{n} \sum_{x \in \mathbb{Z}} v^2(x) < \infty \). Then, there exists \( C > 0 \) such that for any \( t > 0 \) and \( \ell = \varepsilon n \):

\[
\mathbb{E}_\rho \left[ \left( \int_0^t \sum_{x \in \mathbb{Z}} v(x) \left\{ \bar{\eta}_{sn}^2(x) \bar{\eta}_{sn}^2(x + 1) - \left( (\bar{\eta}_{sn}^2(x))^2 - \frac{\chi(\rho)}{\ell} \right) \right\} ds \right)^2 \right] \leq \frac{1}{n} \sum_{x \neq -1} v^2(x),
\]

where

\[
\bar{\eta}^\ell(x) = \frac{1}{\ell} \sum_{y=x+1}^{x+\ell} \bar{\eta}(y).
\]
On the universality of KPZ: exclusion processes

• Let $r : \Omega \rightarrow \mathbb{R}$ be a local function that satisfies:

[i] There exists $\varepsilon_0 > 0$ such that $\varepsilon_0 < r(\eta) < \varepsilon_0^{-1}$ for any $\eta \in \Omega$.

[ii] For any $\eta$, $\xi$ such that $\eta(x) = \xi(x)$ for $x \neq 0, 1$, then $r(\eta) = r(\xi)$.

[iii] Gradient condition. There exists $\omega : \Omega \rightarrow \mathbb{R}$ such that

$$r(\eta)(\eta(1) - \eta(0)) = \tau_1 \omega(\eta) - \omega(\eta),$$

for any $\eta \in \Omega$. 
On the universality of KPZ: zero-range processes

- $\eta_t$ a Markov process with space state $\Omega := \mathbb{N}^\mathbb{Z}$.

- the jump rate from $x$ only depends on the number of particles at $x$ and is given by a function $g : \mathbb{N}_0 \to \mathbb{R}_+$ such that $g(0) = 0$, $g(k) > 0$ for $k \geq 1$ and $g$ is Lipschitz: $\sup_{k \geq 0} |g(k + 1) - g(k)| < \infty$.

As examples:

- If $g$ is Lipschitz and there exists $x_0$ and $\varepsilon_0 > 0$ such that $g(x + x_0) - g(x) \geq \varepsilon_0$ for all $x \geq 0$.
- If $g$ is sublinear, that is $C^{-1}x^\gamma \leq g(x + 1) - g(x) \leq Cx^\gamma$ for $0 < \gamma < 1$ and $C > 0$.
- If $g(x) = 1_{x \geq 1}$. 
On the universality of KPZ: kinetically constrained exclusion processes

- $\eta_t$ is a Markov process with space state $\Omega = \{0, 1\}^\mathbb{Z}$.

- here particles more likely hop to unoccupied nearest-neighbor sites when at least $m - 1 \geq 1$ other neighboring sites are full.

- for $m = 2$, the jump rate to the right is given by:

  $$\eta(x)(1 - \eta(x + 1))\left[\eta(x - 1) + \eta(x + 2) + \frac{\theta}{2n}\right]$$

  and the jump rate to the left is given by

  $$\eta(x + 1)(1 - \eta(x))\left[\eta(x - 1) + \eta(x + 2) + \frac{\theta}{2n}\right].$$
References


3. Franco, G., Simon: *Crossover to the Stochastic Burgers equation for the WASEP with a slow bond*, to appear in CMP.


THANK YOU!