Course on Kipnis Landim

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February 27, 2020

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Invariant measures, Reversibility and Adjoint processes

In the same way that we considered invariant probability measures for discrete time Markov chains, we may investigate invariant measures for continuous time processes, i.e, probability measures μ such that $\mu P_t = \mu$ for every $t \geq 0$.

Proposition

A probability measure μ is invariant if and only if $\mu L = 0$.

Proof.
Of course, if $\mu P_t = \mu$ for every $t \geq 0$, taking the time derivative at $t = 0$, we obtain that $\mu L = 0$.

Conversely, if $\mu L = 0$, by Trotter-Kato formula $P_t = \lim_{t \to 0} \left(I + \frac{t}{n} L \right)^n$, we have that

$$\begin{aligned}
\mu P_t &= \lim_{t\to 0} \mu \left(I + \frac{t}{n} L \right)^n \\
&= \lim_{t\to 0} \mu \left(I + \frac{t}{n} L \right) \left(I + \frac{t}{n} L \right)^{n-1} \\
&= \lim_{t\to 0} \mu \left(I + \frac{t}{n} L \right)^{n-1} \quad \text{doing this sucessively,} \\
&= \mu.
\end{aligned} \tag{1}$$

□

Corollary
A probability measure μ *is invariant if and only if the probability measure*

$$\nu(x) = \frac{\mu(x)\lambda(x)}{\sum_{y \in E} \mu(y)\lambda(y)}$$

is invariant for the discrete time Markov chain with transition probability $p(\cdot,\cdot)$*.*

Proof.
The identity $\mu L = 0$ can be rewritten as

$$\sum_{x \in E} \mu(x)\lambda(x)p(x,y) = \mu(y)\lambda(y)$$

for every $y \in E$. □

It follows from the previous result that under the assumption of indecomposability, that is in general easy to verify, we have the existence of a unique invariant probability measure when E is finite. In the countable case it will also be necessary to check conditions that guaranty the positive recurrence. Most of the time, however, it will be important to characterize more or less explicitly the invariant measures. In order to do it we have to solve the $|E|$ linear equations:

$$\sum_{x \in E} \mu(x)\lambda(x)p(x,y) = \mu(y)\lambda(y)$$

for every $y \in E$. It is stronger but easier to try to solve the system with $|E|^2$ equations

$$\mu(x)\lambda(x)p(x,y) = \mu(y)\lambda(y)p(y,x)$$

known to the physicists as the **detailed balance** condition.

It is clear that a summable solution μ of these equations is automatically an invariant probability measure if properly renormalized. In particular, it is the invariant probability measure if the chain is indecomposable. Furthermore, an invariant measure satisfying the detailed balance conditions possesses certain special properties. To state them we need to introduce some notation.

For an invariant probability measure μ on E, denote by $L^2(\mu)$ the space of the square integrable functions with respect to μ. We extend the operators P_t and L, originally defined on $C_b(E)$, to the space $L^2(\mu)$: for $f \in L^2(\mu)$, we denote by $P_t f$, Lf the functions defined in previous sections.

Remark
L is a bounded operator on $L^2(\mu)$:

$$\langle \mu, [Lf]^2 \rangle \leq 4\overline{\lambda}^2 \langle \mu, f^2 \rangle.$$

Proof.

$$\begin{aligned}
\langle \mu, [Lf]^2 \rangle &= \sum_{x \in E} \mu(x)(Lf(x))^2 \\
&= \sum_{x \in E} \mu(x) \left(\sum_{y \in E} \lambda(x) p(x,y)[f(y) - f(x)] \right)^2 \\
&\leq \sum_{x \in E} \mu(x) \sum_{y \in E} \lambda(x)^2 p(x,y)^2 [f(y) - f(x)]^2 \\
&\leq \sum_{x \in E} \mu(x) \sum_{y \in E} \lambda(x)^2 p(x,y) [2f(y)^2 + 2f(x)^2] \\
&\leq 2 \sum_{x,y \in E} \mu(x)\lambda(x)^2 p(x,y) f(y)^2 + 2 \sum_{x,y \in E} \mu(x)\lambda(x)^2 p(x,y) f(x)^2
\end{aligned} \tag{2}$$

□

Moreover, it follows from the estimates that we saw in section 3 that for every function f in $L^2(\mu)$ we have that:

$$\lim_{t \to 0} \langle \mu, \{t^{-1}[P_t f - f] - Lf\}^2 \rangle = 0. \tag{3}$$

Proposition

The probability measure μ satisfies the detailed balance equation if and only if the operators P_t are self-adjoint in $L^2(\mu)$, i.e., if and only if for every $t \geq 0$, $f, g \in L^2(\mu)$,

$$\sum_{x \in E} \mu(x) f(x)(P_t g)(x) = \sum_{x \in X} \mu(x) g(x)(P_t f)(x)$$

or, briefly,

$$\langle f, P_t g \rangle_\mu = \langle P_t f, g \rangle_\mu . \tag{4}$$

Here $\langle \cdot, \cdot \rangle_\mu$ stands for the inner product in $L^2(\mu)$.

Proof.
Fix two sites, x, y set $f = \mathbf{1}\{x\}$, $g = \mathbf{1}\{y\}$ and take the time derivative at $t = 0$ in identity (4) to obtain that

$$\mu(x)L(x,y) = \mu(y)L(y,x)$$

which is the detailed balance condition. Inversely, it follows from the previous identity that

$$\langle f, Lg\rangle_\mu = \langle Lf, g\rangle_\mu$$

for every f, g in $L^2(\mu)$. It remains to recall Trotter-Kato formula to conclude the proof. $\square$

A probability measure satisfying the detailed balance conditions is said to be reversible. The previous result states therefore that a probability measure is reversible if and only if the generator is self adjoint in $L^2(\mu)$. We shall now look for conditions that guarantee that the adjoint of the generator L in $L^2(\mu)$ is also a generator.

Proposition

Let μ be a probability measure. The adjoint of L in $L^2(\mu)$, denoted by $L^\star$, is a generator if and only if μ is invariant. In this case $P_t^\star = e^{tL^\star}$ is also the adjoint of P_t in $L^2(\mu)$ and the semigroup $P_t^\star$ is characterized by the pair $(\lambda^\star, p^\star)$ given by

$$\begin{cases} \lambda^\star(x) = \lambda(x), \forall x \in E \\ p^\star(x,y) = \frac{\lambda(y)\mu(y)p(y,x)}{\lambda(x)\mu(x)}, \forall x, y \in E. \end{cases}$$

Moreover,

$$\langle f, Lg \rangle_\mu = \langle L^\star f, g \rangle_\mu, \tag{5}$$

for every $f, g \in L^2(\mu)$.

Proof

A simple computation shows that the adjoint $L^\star$ of L in $L^2(\mu)$ is given by the formula

$$\mu(x)L^\star(x,y) = \mu(y)L(y,x). \tag{6}$$

In particular, $L^\star$ satisfies always the first the first two properties of generators:

$$\begin{cases} L^\star(x,x) = -\lambda(x) < 0, \text{ for } x \in E \\ L^\star(x,y) \geq 0, \text{ for } x \neq y. \end{cases} \tag{7}$$

Therefore, $L^\star$ is a generator if and only if

$$\sum_{y\in E} L^\star(x,y) = 0$$

for every $x \in E$ and hence if and only if μ is invariant because the explicit expression (6) permits to rewrite the previous sum as

$$\frac{1}{\mu(x)} \sum_{y\in E} \mu(y)L(y,x).$$

On the other hand, the Trotter-Kato formula shows that the semigroup $P_t^\star$ associated to the generator $L^\star$ is the adjoint of the semigroup P_t in $L^2(\mu)$.

Finally, we have already seen in the first part of the proof that the jump rate λ and $\lambda^\star$ coincide. The explicit formula (6) permits than to compute $p^\star(x, y)$. On the other hand, formula (5) follows from the identity $\mu(x)\lambda(x)p(x, y) = \mu(y)\lambda(y)p^\star(y, x)$ and a change of variables:

$$\langle f, Lg\rangle_\mu = \sum_{x,y} \mu(x)\lambda(x)p(x, y)f(x)g(y) - \sum_{x\in E} \mu(x)\lambda(x)f(x)g(x).$$

The first term on the right hand side can be written as

$$\sum_{x,y} \mu(y)\lambda(y)p^\star(y, x)f(x)g(y) = \sum_{x,y\in E} \mu(x)\lambda(x)p^\star(x, y)f(y)g(x)$$

from the previous two identities it is easy to conclude the proof of the proposition. □

The process $P_t^\star$, that is defined without ambiguity when the original process admits a unique invariant measure, is called the adjoint process. The adjoint process is closely connected to the process reversed in time. This is the content of the next proposition.

Proposition

If the semigroup $P_t^\star$ is the adjoint of the semigroup P_t with respect to the invariant probability measure μ, then for every $n > k \geq 0$, every sequence of times $0 \leq t_1 < \ldots < t_n$ and every sequence of bounded functions $\{f_j; 1 \leq j \leq n\}$,

$$\begin{aligned} &E_\mu[f_1(X_{t_1}) \ldots f_n(X_{t_n})] \\ &= \sum_{x \in E} \mu(x) f_k(x) E_x[f_{k+1}(X_{t_{k+1}-t_k}) \ldots f_n(X_{t_n - t_k})] \times \\ &\times E_x^\star[f_k(X_{t_k - t_{k-1}}) \ldots f_1(X_{t_k - t_1})], \end{aligned}$$

where $E_x^\star$ stands for the expectation with respect to the Markov chain with transition probability $P_t^\star$ starting from x.

Proof.
The proof is straightforward. We just have to apply successively the identity

$$\mu(x)P_t(x,y) = \mu(y)P_t^\star(y,x).$$

$\square$

We shall sometimes consider the symmetric part, denoted by S, of the generator L in $L^2(\mu)$. It is given by

$$S = 2^{-1}(L + L^\star).$$

A simple computation shows that the symmetric part S is itself a generator characterized by the parameters $\lambda^\star$ and $p^\star$ given by

$$\lambda^s(x) = \lambda(x), \forall x \in E,$$
$$p^s(x,y) = \frac{1}{2}\Big[p(x,y) + \frac{\lambda(y)\mu(y)p(y,x)}{\lambda(x)\mu(x)}\Big].$$

Furthermore, S satisfies the detailed balanced condition:

$$\mu(x)S(x,y) = \mu(y)S(y,x).$$

Some Martingales in the context of Markov process

Some Martingales in the context of Markov process

The purpose of this section is to introduce a class of martingales in the context of Markov processes. Consider a bounded function $F : \mathbb{R}_+ \times E \to \mathbb{R}$ smooth in the first coordinate uniformly over the second: for each $x \in E$, $F(\cdot, x)$ is twice continuously differentiable and there exists a finite constant C such that

$$\sup_{(s,x)} \left|(\partial_s^1 F)(s,x)\right| \leq C, \quad \sup_{(s,x)} \left|(\partial_s^2 F)(s,x)\right| \leq C. \tag{8}$$

In this formula, $(\partial_s^1 F)(s,x)$ stands for the first time derivative of $F(\cdot, x)$ and $(\partial_s^2 F)(s,x)$ stands for the second time derivative of $F(\cdot, x)$.

To each function F satisfying assumption (8), define $M^F(t)$ and $N^F(t)$ by

$$M^F(t) = F(t, X_t) - F(0, X_0) - \int_0^t (\partial_s + L)F(s, X_s)ds,$$

$$N^F(t) = \big(M^F(t)\big)^2 - \int_0^t \Big\{\big(LF(s, X_s)\big)^2 - 2F(s, X_s)LF(s, X_s)\Big\}ds.$$

Lemma

Denote by $\{\mathcal{F}_t, t \geq 0\}$ *the filtration induced by the Markov process:* $\mathcal{F}_t = \sigma(X_s, s \leq t)$*. Then the processes* $M^F(t)$ *and* $N^F(t)$ *are* $\mathcal{F}_t$*-martingales.*

Proof

We start showing that $M^F(t)$ is a martingale. Fix $0 \le s < t$. We need to check that

$$E_x\Big[M^F(t)|\mathcal{F}_s\Big] = M^F(s).$$

In other words,

$$\begin{aligned}
& E_x\Big[F(t,X_t) - F(0,X_0) - \int_0^t (\partial_r + L)F(r,X_r)dr|\mathcal{F}_s\Big] \\
= \quad & F(s,X_s) - F(0,X_0) - \int_0^s (\partial_r + L)F(r,X_r)dr \\
\Longrightarrow \quad & E_x\Big[F(t,X_t)|\mathcal{F}_s\Big] = F(s,X_s) + \int_s^t E_x\Big[(\partial_r + L)F(r,X_r)|\mathcal{F}_s\Big]dr. \qquad (9)
\end{aligned}$$

For each $r \geq 0$, denote by $F_r : E \to \mathbb{R}$ (resp. $F'_r : E \to \mathbb{R}$) the function that at x takes the value $F(r,x)$ (resp. $\partial_r F(r,x)$). By the Markov property and a change of variables in the integral, the previous identity is reduced to

$$(P_{t-s}F_t)(X_s) = F_s(X_s) + \int_0^{t-s} \{(P_r F'_{s+r})(X_s) + P_r L F_{s+r}(X_s)\} \, dr. \tag{10}$$

Since for $t = s$ this identity is trivially satisfied, we just need to check that the time derivative of both expressions are equals, i.e., that

$$\partial_t(P_{t-s}F_t)(x) = (P_{t-s}F_t')(x) + (P_{t-s}LF_t)(x) \tag{11}$$

for every $x \in E$ and $0 \leq s < t$.

To prove this identity we compute the left hand side. Fix $h > 0$ and rewrite the difference

$$\frac{(P_{t-s+h}F_{t+h})(x) - (P_{t-s}F_t)(x)}{t}$$

as

$$h^{-1}E_x\Big[F_{t+h}(X_{t-s+h}) - F_t(X_{t-s+h})\Big] + h^{-1}E_x\Big[F_t(X_{t-s+h}) - F_t(X_{t-s})\Big]. \tag{12}$$

The first expression is equal to

$$\frac{1}{h}\int_t^{t+h} dr\, E_x\Big[F_r'(X_{t-s+h}) - F_t'(X_{t-s+h})\Big] + E_x\Big[F_t'(X_{t-s+h}) - F_t'(X_{t-s})\Big] + E_x\Big[F_t'(X_{t-s})\Big].$$

Since by assumption, $(\partial_r F)(\cdot, x)$ is Lipschitz continuous uniformly on x, the first term vanishes as $h \to 0$. The second term also vanishes as $h \to 0$ because the semigroup P_t is continuous. Therefore, as $h \to 0$, the first term in (12) converges to $(P_{t-s} F_t')(x)$. The second expression in (12) is equal to

$$\frac{1}{h} \int_{t-s}^{t-s+h} dr \, E_x \Big[L F_t(X_r) \Big]$$

that converges, as $h \to 0$, to $(P_{t-s} L F_t)(x)$. This proves that $M^F(t)$ is a martingale.

We show now that $N^F(t)$ is a martingale. A simple computation and the first part of the lemma show that $M^F(t)^2$ is equal to

$$F^2(t,X_t) - 2F(t,X_t)\int_0^t ds\,(\partial_s + L)F(s,X_s) + \left(\int_0^t ds\,(\partial_s + L)F(s,X_s)\right)^2 \tag{13}$$

plus a martingale term. Since $F^2(t,X_t) - \int_0^t ds\,(\partial_s + L)F^2(s,X_s)$ is a martingale, $F^2(t,X_t)$ is equal to a martingale added to

$$\int_0^t ds\,(\partial_s + L)F^2(s,X_s). \tag{14}$$

The second expression in (13) can be rewritten as

$$\begin{aligned}-2M^F(t)\int_0^t ds\,(\partial_s + L)F(s,X_s) - 2F(0,X_0)\int_0^t ds\,(\partial_s + L)F(s,X_s) \\ - \left(\int_0^t ds\,(\partial_s + L)F(s,X_s)\right)^2.\end{aligned} \tag{15}$$

By Ito's formula, the first term in this expression is equal to a martingale added to

$$
\begin{aligned}
&- 2 \int_0^t ds \, F(s, X_s)(\partial_s + L)F(s, X_s) \\
&+ 2 \int_0^t ds \left(\int_0^s dr \, (\partial_r + L)F(r, X_r) \right) (\partial_s + L)F(s, X_s).
\end{aligned}
$$

An integration by parts shows that the second term of this formula cancels with the third term of (15). The remaining expression added to (14) is just the integral term we need to subtract in order to turn $(M^F(t))^2$ in a martingale. This conclude the proof of the lemma. □

Estimates on the variance of additive functionals of Markov processes

Estimates on the variance of additive functionals of Markov processes

Consider a Markov Process with generator L satisfying the assumptions of section 3 and having an invariant measure denoted by π. Let S be the symmetric part of the generator L: $S = 2^{-1}(L + L^\star)$. For a function $f \in L^2(\pi)$, denote by $||f||_1$ its $\mathcal{H}_1$ norm defined by

$$||f||_1^2 = \langle f, (-S)f \rangle_\pi$$

and notice that $||f||_1^2 = -\langle f, Lf \rangle_\pi$. Recall from section 4 that we denote by $p^s(x, y)$ the transition probability associated to the symmetric part of the generator. With this notation,

$$||f||_1^2 = \frac{1}{2} \sum_{x,y \in E} \pi(x)\lambda(x)p(x, y)\{f(y) - f(x)\}^2.$$

In particular, by Schwarz inequality, the $\mathcal{H}_1$ norm $||f||_1^2$ of any function f in $L^2(\pi)$ is bounded above by $2||\lambda||_\infty ||f||_0^2$, provided $||f_0||$ stands for the $L^2(\pi)$ norm of f.

Let $\mathcal{H}_1$ be the Hilbert space generated by $L^2(\pi)$ and the inner product $\langle f, g\rangle_1 = \langle f, (-S)g\rangle_\pi : \mathcal{H}_1 = \overline{L^2(\pi)|_{\mathcal{N}}}$, if $\mathcal{N}$ stands for the kernel of the inner product $\langle f, g\rangle_1$ (In the context of this chapter, $\mathcal{N}$ is the space generated by constant functions because we assumed the skeleton discrete time Markov chain to be indecomposable). Since $||f||_1^2 \leq 2||\lambda||_\infty ||f||_0^2$, $L^2(\pi) \subset \mathcal{H}_1$.

Denote by $||\cdot||_{-1}$ the dual norm of $\mathcal{H}_1$ with respect to $L^2(\pi)$: for $f \in L^2(\pi)$, let

$$||f||_{-1}^2 = \sup_{g \in L^2(\pi)} \{2\langle f, g\rangle_\pi - ||g||_1^2\}. \tag{16}$$

We claim that $||f||_{-1}^2 \geq (2||\lambda||_\infty)^{-1}||f||_0^2$ for each function $f \in L^2(\pi)$. Indeed, since $||g||_1^2 \leq 2||\lambda||_\infty ||g||_0^2$, the supremum on the right side of (16) is bounded below by

$$\sup_{g \in L^2(\pi)} \{2\langle f, g\rangle_\pi - |2||\lambda||_\infty ||g||_0^2\}.$$

By Schwarz inequality, this supremum is equal to $(2||\lambda||_\infty)^{-1}||f||_0^2$.

Denote by $\mathcal{H}_{-1}$ the subset of $L^2(\pi)$ of all functions with finite $||\cdot||_{-1}$ norm. It follows from the definition of the $\mathcal{H}_{-1}$ norm that

$$2\langle f, g\rangle_\pi \leq \frac{1}{A}||f||_{-1}^2 + A||g||_1^2 \tag{17}$$

for every $f \in \mathcal{H}_{-1}, g \in L^2(\pi), A > 0$. Since $L^2(\pi)$ is dense in $\mathcal{H}_1$, this inequality may be extended to functions $g\mathcal{H}_1$. Moreover, we clain that $SL^2(\pi) = \{Sf, f \in L^2(\pi)\}$ is contained in $\mathcal{H}_{-1}$ and

$$||Sf||_{-1}^2 = ||f||_1^2, \forall f \in L^2(\pi).$$

Indeed, fix $f \in L^2(\pi)$. A simple computation shows that

$$2\langle g, Sf\rangle_\pi \leq \langle g, (-S)g\rangle_\pi + \langle f, (-S)f\rangle_\pi = ||g||_1^2 + ||f||_1^2$$

for every $g \in L^2(\pi)$. In particular, $||Sf||_{-1} \leq ||f||_1$. Taking $g = -f$ in the supremum that defines $||f||_1^2$ we deduce the reverse inequality.

We are now ready to state the main result of this section.

Proposition
For each function $g \in \mathcal{H}_{-1}$ and $t > 0$,

$$E_\pi\Big[\Big(\frac{1}{\sqrt{t}}\int_0^t g(X_s)ds\Big)^2\Big] \leq 20||g||_{-1}^2.$$

Proof

Since L is a generator, for every $\gamma > 0$, $(\gamma - L)^{-1}$ is a bounded operator in $L^2(\pi)$. In particular, since $\mathcal{H}_{-1} \subset L^2(\pi)$, there exists f_γ in $L^2(\pi)$ such that

$$\gamma f_\gamma - L f_\gamma = g.$$

Taking on both sides of this equation the inner product with respect to f_γ and applying Schwartz inequality (17), we obtain that

$$\gamma \langle f_\gamma, f_\gamma \rangle_\pi \leq ||g||_{-1}^2 \text{ and } ||f_\gamma||_1^2 \leq ||g||_{-1}^2. \tag{18}$$

For $\gamma > 0$ consider the process $M_\gamma(t)$ defined by

$$M_\gamma(t) = f_\gamma(X_t) - f_\gamma(X_0) - \int_0^t L f_\gamma(X_s)\, ds.$$

By Lemma 5.1, $M_\gamma(t)$ is a martingale if f_γ is bounded. Approximating f_γ, that belongs to $L^2(\pi)$, by bounded functions we deduce that $M_\gamma(t)$ is a martingale in $L^2(\pi)$. With this notation we may rewrite the expectation appearing in the statement of the lemma as

$$\frac{1}{t} E_\pi \Big[\Big(M_\gamma(t) - f_\gamma(X_t) + f_\gamma(X_0) + \int_0^t \gamma f_\gamma(X_s)\, ds \Big)^2 \Big].$$

Since π is an invariant measure for the Markov process, by Schwartz inequality, this expression is bounded above by

$$\frac{4}{t} \Big\{ [2 + (\gamma t)^2] \langle f_\gamma, f_\gamma \rangle_\pi + E_\pi \big(M_\gamma(t)^2 \big) \Big\}.$$

By Lemma 5.1 the quadratic variation of the martingale $M_\gamma(t)$ is equal to the time integral of

$$L f_\gamma(X_s)^2 - 2 f_\gamma(X_s) L f_\gamma(X_s).$$

In particular, since the probability measure π is invariant, the expectation of $M_\gamma(t)^2$ is equal to $2t||f_\gamma||_1^2$ because $-\langle f, Lf\rangle_\pi = ||f||_1^2$ for every $f \in L^2(\pi)$. Therefore, the expectation appearing on the statement of the lemma is less than or equal to

$$\begin{aligned} &\frac{4}{t}\Big\{[2+(\gamma t)^2]\langle f_\gamma, f_\gamma\rangle_\pi + 2t||f_\gamma||_1^2\Big\} \\ &\leq \Big\{4[2+(\gamma t)^2](t\gamma)^{-1} + 8\Big\}||g||_{-1}^2 \end{aligned} \tag{19}$$

in virtue of (18). To conclude the proof of the proposition, it remains to choose $\gamma = t^{-1}$. □

The Feynman-Kac Formula

The Feynman-Kac Formula

Consider a bounded function $V : \mathbb{R}_+ \times E \to \mathbb{R}$ satisfying assumption (8) and a bounded function $F_0 : E \to \mathbb{R}$. Fix $T > 0$ and denote by $F : [0, T] \times E \to \mathbb{R}$ the solution of the differential equation

$$\begin{cases} (\partial_t u)(t,x) = (Lu)(t,x) + V(T-t,x)u(t,x), \\ u(0,x) = F_0(x). \end{cases} \tag{20}$$

Proposition

The solution F has the following stochastic representation:

$$F(T,x) = E_x\left[e^{\int_0^T V(s,X_s)ds} F_0(X_T)\right].$$

Proof
Consider the process $\{A_t, 0 \le t \le T\}$ given by

$$A_t = F(T-t, X_t)\exp\left\{\int_0^T V(s, X_s)ds\right\}.$$

By Lemma 5.1, A_t can be rewritten as

$$\left\{M_0^F(t) + \int_0^t ds(\partial_s + L)F(T-s, X_s)\right\}\exp\left\{\int_0^t V(s, X_s)ds\right\},$$

where $M_0^F(t)$ is the martingale

$$F(T-t, X_t) - \int_0^t (\partial_s + L)F(T-s, X_s)ds.$$

Ito's formula now gives that

$$A_t - \int_0^t ds\, e^{\int_0^s V(r, X_r)\, dr}\{F(T-s, X_s)V(s, X_s) + (\partial_s + L)F(T-s, X_s)\}$$

is a martingale.

Since F is the solution of (20), the integral term vanishes showing that A_t is martingale. In particular,

$$E_x[A_T] = E_x[A_0],$$

which proves the Proposition. $\square$

Remark
The proof of the previous Proposition shows that

$$e^{\int_0^t V(r,X_r)dr} F(X_t) - \int_0^t ds\, e^{\int_0^s V(r,X_r)dr} \{(LF(X_s) + V(s,X_s)F(X_s)\} \tag{21}$$

is a martingale for each bounded function $F : E \to \mathbb{R}$.

For $t \geq 0$, denote by $L_t : E \times E \to \mathbb{R}$ the operator defined by

$$L_t(x,y) = L(x,y) + V(t,y)\delta_{x,y}$$

where $\delta_{x,y}$ stands for the delta of Kronecker. Denote furthermore, for $0 \leq s \leq t$, by $P^V_{s,t} : E \times E \to \mathbb{R}_+$ the operator given by

$$P^V_{s,t}(x,y) = E\left[e^{\int_s^t V(r,X_r)dr}\mathbf{1}\{X_t = y\}|X_s = x\right]. \qquad (22)$$

A simple computation, relying on Markov property, permits to rewrite $P^V_{s,t}(x,y)$ as

$$P^V_{s,t}(x,y) = E_x\left[e^{\int_0^{t-s} V(s+r,X_r)dr}\mathbf{1}\{X_{t-s} = y\}\right].$$

The collection $\{P^V_{s,t} : 0 \le s \le t\}$ can be extended to act on bounded functions:

$$(P^V_{s,t} f)(x) = E_x\left[e^{\int_0^{t-s} V(s+r, X_r)\,dr} f(X_{t-s})\right].$$

Property (21) applied to the bounded function $F(z) = \mathbf{1}\{z = y\}$ and a simple computation shows that $\{P^V_{s,t}, 0 \le s \le t\}$ is a semigroup associated to the operator $L_t : P^V_{s,t} P^V_{t,u}$ for all $0 \le s \le t \le u$ and

$$\begin{cases} (\partial_t P^V_{s,t})(x,y) = (P^V_{s,t} L_t)(x,y), \\ P^V_{s,s}(x,y) = \delta_{x,y} \end{cases} \tag{23}$$

for $t \ge s \ge 0$.

The arguments presented in Section 3 permit to prove also the Chapman-Kolmogorov equations:

$$\begin{cases} (\partial_s P^V_{s,t})(x,y) = -(L_s P^V_{s,t})(x,y), \\ P^V_{t,t}(x,y) = \delta_{x,y} \end{cases} \tag{24}$$

for $0 \leq s \leq t$.

Assume now that L is a reversible generator with respect to an invariant measure state ν. Since V is bounded, $L_t = L + V(t,\cdot)$ is also a symmetric operator in $L^2(\nu)$.

Denote by Γ_t the largest eigenvalue of $L + V_t$:

$$\Gamma_t = \sup_{||f||_2=1} \left\{ \langle V_t, f^2 \rangle_\nu + \langle Lf, f \rangle_\nu \right\}.$$

By definition of the semigroup $\{P^V_{s,t},\, 0 \le s \le t\}$,

$$E_\nu\left[e^{\int_0^t V(r,X_r)dr}\right] = \langle P^V_{0,t}\mathbf{1}, \mathbf{1}\rangle_\nu.$$

On the other hand, by the first Chapman-Kolmogorov equation, for $0 \le s \le t$,

$$\frac{d}{ds}\langle P^V_{s,t}\mathbf{1}, P^V_{s,t}\mathbf{1}\rangle_\nu = -2\langle L_s P^V_{s,t}\mathbf{1}, P^V_{s,t}\mathbf{1}\rangle_\nu$$

because L_s is a symmetric operator. By definition of Γ_s, this expression is bounded below by

$$-2\Gamma_s\langle P^V_{s,t}\mathbf{1}, P^V_{s,t}\mathbf{1}\rangle_\nu.$$

Therefore, by Gronwall inequality and since $P^V_{t,t}$ is the identity,

$$\langle P^V_{0,t}\mathbf{1}, P^V_{0,t}\mathbf{1}\rangle_\nu \le \exp\left\{\int_0^t \Gamma_s ds\right\}.$$

Since, by Schwarz inequality,

$$\langle P_{0,t}^V \mathbf{1}, \mathbf{1}\rangle_\nu \leq \langle P_{0,t}^V \mathbf{1}, P_{0,t}^V \mathbf{1}\rangle_\nu^{1/2},$$

we have proved the following lemma:

Lemma
Assume that the Markov process is reversible with respect to an invariant probability measure ν. Let $V : \mathbb{R}_+ \times E \to \mathbb{R}$ be a bounded function. For each $t \geq 0$, denote by Γ_t the largest eigenvalue of the operator $L + V(t, \cdot)$. Then,

$$E_\nu\Big[exp\{\int_0^t V(r, X_r)dr\}\Big] \leq exp\{\int_0^t \Gamma_s ds\}. \tag{25}$$

Remark
In the case where ν is only an invariant measure, it is possible to prove that (25) remains in force provided Γ_s is the largest eigenvalue of $S = \frac{1}{2}(L + L^\star)$, the symmetric part of $L \in L^2(\nu)$.

The Feynman-Kac formula presented in Proposition 6 permits to obtain an explicit formula for the Radon-Nikodyn derivative of a time inhomogeneous Markov process with respect to another, generalizing Proposition 2.6 to the inhomogeneous case.

Fix a function $F : \mathbb{R}_+ \times E \to \mathbb{R}$ satisfying assumptions (5.1). Denote by $\mathbb{M}^F(t)$ the process defined by

$$\mathbb{M}^F(t) = \exp\{F(t, X_t) - F(0, X_0) - \int_0^t ds\, e^{-F(s,X_s)}(\partial_s + L)e^{F(s,X_s)}\}.$$

We claim that $\mathbb{M}^F(t)$ is a mean 1 positive martingale. Indeed, fix x_0 in E and define $V, H : \mathbb{R}_+ \times E \to \mathbb{R}$ by

$$\begin{aligned} V(t, x) &= -\exp\{-F(t, x)\}(\partial_s + L) \exp\{F(t, x)\}, \\ H(t, x) &= \exp\{F(t, x) - F(0, x_0)\}. \end{aligned} \tag{26}$$

It follows from the proof of Proposition 6 that

$$\mathbb{M}^F(t) = H(t, X_t) \exp\left\{\int_0^t V(s, X_s) ds\right\}$$

is a martingale. Since this martingale is equal to 1 P_{x_0} almost surely at time 0, it is a mean 1 positive martingale.

Fix a time $T > 0$ and for each x_0 in E, define on $\mathcal{F}_T$ the probability measure $P^F_{x_0}$ by

$$E^F_{x_0}[G] = E_{x_0}[G\,\mathbb{M}^F(T)] \tag{27}$$

for all bounded $\mathcal{F}_T$-measurable functions G. A simple computation shows that the conditional expectation is

$$E^F_{x_0}[G\,|\mathcal{F}_s] = \frac{1}{\mathbb{M}^F(s)} E_{x_0}[G\,\mathbb{M}^F(T)|\mathcal{F}_s].$$

For $0 \leq s \leq t \leq T$, define the functions $Q^F_{s,t} : E \times E \to \mathbb{R}_+$ by

$$Q^F_{s,t}(x, y) = P^F_{x_0}\{X_t = y \,|\, X_s = x\}.$$

Recall the definition of the function V introduced before and of the semigroup $\{P^V_{s,t},\, 0 \leq s \leq t\}$ given by (22). It follows from the formula for the conditional expectation that

$$Q^F_{s,t}(x, y) = P^V_{s,t}(x, y) \exp\{F(t, y) - F(s, x)\}.$$

We claim that $\{Q^F_{s,t}, 0 \leq s \leq t \leq T\}$ is a semigroup of transition probabilities. Indeed, it is clear that $Q^F_{s,t}(x, y) \geq 0$ for all x, y in E. It follows from the explicit formula for the conditional expectation that $\sum_{y \in E} Q^F_{s,t}(x, y) = 1$ for all $0 \leq s \leq t \leq T$ and $x \in E$.

Finally, by definition of $Q^F_{s,t}$ and because $\{P^V_{s,t}, 0 \le s \le t\}$ is a semigroup,

$$\sum_{z \in E} Q^F_{s,t}(x,z) Q^F_{t,u}(z,y) = Q^F_{s,u}(x,y)$$

for all $s \le t \le u$ and all x, y in E. This proves that $\{Q^F_{s,t\,0 \le s \le t \le T}\}$ is a semigroup of transitions probabilities. We now compute the generator associated to this semigroup. Since $Q^F_{s,t}(x,y) = P^V_{s,t}(x,y) \exp\{F(t,y) - F(s,x)\}$, by the second Chapman-Kolmogorov equations (7.4),

$$\begin{aligned} \partial_t Q^F_{s,t}(x,y) &= \partial_t \left\{ P^V_{s,t}(x,y) \exp\{F(t,y) - F(s,x)\} \right\} \\ &= e^{\{F(t,y) - F(s,x)\}} \left\{ (P^V_{s,t} L_t)(x,y) + P^V_{s,t}(x,y)(\partial_t F(t,y)) \right\}. \end{aligned}$$

The explicit formula for L_t and some elementary computations lead to the formula

$$\partial_t Q^F_{s,t}(x,y) = (Q^F_{s,t} L^F_t)(x,y),$$

where

$$(L^F_t)(x,y) = L(x,y)e^{\{F(t,y)-F(t,x)\}} - \delta_{x,y} e^{\{F(t,y)\}} L e^{\{F(t,y)\}}.$$

In particular, for any bounded function $H : E \to \mathbb{R}$,

$$(L^F_t H)(x) = \sum_{y\in E} \lambda(x) p(x,y) \exp\{F(t,y) - F(t,x)\}[H(y) - H(x)]. \tag{28}$$

We summarize in the next proposition the result we just proved.

Proposition

Fix a function $F : \mathbb{R}_+ \times E \to \mathbb{R}$ *satisfying assumptions* (8). *Fix a time* $T > 0$ *and for each* $x_0 \in E$, *define on* $\mathcal{F}_T$ *the probability measure* $P^F_{x_0}$ *given by* (27). *Then, under* $P^F_{x_0}$, X_t *is a time inhomogeneous Markov process with generator* $\{L^F_t, 0 \leq t \leq T\}$ *given by* (28) *and starting from* x_0.

Obrigada!